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Characterization of the time irreversibility of economic time-series

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Characterization of the Time Irreversibility of Economic Time Series

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Table of Contents

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1. Introduction

If the probabilistic structure of a time series going forward in time is identical to that in reverse time, the series is time reversible. If the series is not time reversible, it is said to be time irreversible. While the notion of time reversibility has a long history in physics and has been developed in the stochastic process literature within the framework of Markov chains, it was first mentioned in statistical time series analysis by Daniels (1946). The first formal statistical definition was given by Brillinger and Rosenblatt (1967, p. 210) about twenty years ago.

The issue of time reversibility is important for economics for both theoretical and empirical reasons. The behavior of key variables in representative members of wide classes of conventional macroeconomic models is time reversible. For example, the cyclical component of output in Lucas's (1973) New Classical model is time reversible. Also, the growth rates of output and other variables in Eichenbaum and Singleton's (1986) Real Business Cycle model are time reversible. Evidence that these time series are time irreversibile would then show that an implication of these models is not satisfied by the data.

Both of these models are members of the class of "Frisch-type" business cycle models. These are models based on the distinction between impulse and propagation mechanisms. Independently and identically distributed shocks provide impulses which affect output through distributed lag relations, the propagation mechanism. This modelling strategy stems from the early work of Frisch (1937) and Slutsky (1933) in which they showed that a linear system of equations driven by random shocks could produce business cycle-like behavior in the sample path of a random variable.[1] [2] Blanchard and Fischer (1989) stress that, while macroeconomists disagree both as to the main sources of these shocks (e.g., real or nominal, demand or supply, stemming from the private sector or from the government) and the exact nature of the propagation mechanism, the Frisch-type approach is currently the dominant one in both theoretical and empirical macroeconomics.

Blatt (1980) recently demonstrated that Frisch-type models are unable to capture cyclical asymmetries; asymmetries due to differences

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1 See Sargent (1979, Chapter 9) for more details.

2 Blanchard and Fischer (1989, p. 311) note that theoretical models often suggest the presence of nonlinearities. They point out, however, that these nonlinearities usually do not play a crucial in the propagation mechanisms.

in the dynamic structure across business cycle expansions and contractions. If a time series is time reversible, it is straightworward to see that the probabilistic structure as the series increases is the same as when the series decreases. Thus, the result that fluctuations in Frisch-type models are symmetric implies that these models are time reversible. In this light, the empirical question of business cycle asymmetry, studied in a line of work opened up by Neftci (1984), is seen to be a question of whether the dynamic behavior of key macroeconomic variables is time reversible. In this dissertation, then, I define business cycle asymmetry as time irreversibility. This provides a unified framework for addressing the issue.

Evidence that key macroeconomic variables have irreversible dynamics would then suggest that a Frisch-type approach might be inappropriate and misleading. For example, in the energy economics literature researchers have begun to address the issue of long-run price asymmetries.[3] This refers to a hysteresis-type phenomenon in which the long-run equilibrium relationship is itself a function of history. I use the term hysteresis in the loose sense of Blanchard and Summers (1986, p. 17), to represent a case in which the degree of

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[3] I am indebted to Professor Dermot Gately for this reference. Details can be found in Sweeney (1986). See Gately and Rappoport (1986) for an empirical application.

dependence on the past is very high. It may be that, for example, after a period of lower prices the system tends toward a higher demand function.[4] In the literature it has been stressed that such long-run asymmetries are incompatible with distributed-lag propagation mechanisms. Hence, if long-run asymmetries are important, incorrect inferences would be drawn from a distributed-lag demand function estimated over a period in which, for example, price reductions follow price increases.

If the rate of adjustment toward long-run equilibrium differs across phases of the business cycle, then conventional forecasting techniques, such as Gaussian ARMA models, will clearly be biased. If movements are slower in expansions than in contractions, the standard time series tools will be biased upwards during expansions and they will under-predict during contractions. It can indeed be shown formally that stationary Gaussian ARMA models are time reversible. Hence, detection of irreversibility in a particular time series implies that the conventional Gaussian ARMA approach is not an

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[4] To his great credit, Georgescu-Roegen (1950) anticipated the current discussion of hysteresis effects in his critique of the conventional static view of the law of demand. He argued that preferences depend upon the economic experience of the individual agent. As such, he claimed that following an initial change in prices, except by chance no new shift in prices alone can bring the consumer back to his original position, since his indifference map has been altered.

appropriate modelling strategy. This point has been strongly emphasized by Tong (1983) and Subba Rao and Gabr (1984). Irreversible behavior would require consideration of alternative time series models capable of capturing this property.

Recent work suggests that this may generate fruitful results. Consider the problem of forecasting, in difference stationary form, quarterly U.S. GNP. In an important paper Potter (1989) reported an improvement, over traditional autoregressive models, of more than 10% in root mean square error using one-step ahead forecasts from a threshold autoregressive (TAR) model. The TAR model consists of a set of autoregressive models, one of which is chosen for any particular period conditional on the state of the system at some lag. Within each regime the model is linear, but the model switches across regimes as some threshold value is passed. If the type of asymmetry mentioned above is indeed present for some business cycle indicator, the TAR approach would be a natural one to adopt (e.g., different models for expansions and contractions).

In this dissertation I introduce a time domain test statistic to identify and characterize time irreversible stationary time series. No other test for time irreversibility has been developed in the literature. The statistic is constructed by taking the difference between two particular bicovariances for a time series. In this sense the test shares some common features with the bispectrum frequency domain test of Hinich (1982)[5]. However, my test is far less data intensive than Hinich's and, perhaps more importantly, can serve as a direct guide to specification of an appropriate time series model. As such this test is in line with the general research program, initiated by Hinich and his colleagues, of trying to detect nonlinear behavior in economic time series[6].

In Chapter 2, I survey the major empirical work done on the business cycle asymmetry hypothesis. In this chapter I also review two related research programs in the time series literature: (1) the BDS test of Brock, Dechert and Scheinkman (1988); and (2) the Hinich's bispectrum test.

In Chapter 3, time reversibility is defined formally and a tool for identifying time irreversible processes, the symmetric-bicovariance function, is introduced. In this chapter it is shown that independently and identically distributed processes and Gaussian ARMA processes are time reversible. Further, by way of simple examples it is shown that time irreversibility can result from two sources: (1) the underlying innovations to the process are drawn from

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5 Hsieh (1988) and LeBaron (1988) also look at sample bicovariances to detect nonlinearities. A recent paper which utilizes the sample bicovariances, but in a different context, is Ramsey and Montenegro (1988).

6 Hinich and Patterson (1985a), Hinich and Patterson (1985b), Ashley, Patterson and Hinich (1986), Hinich and Patterson (1987).

a non-symmetric probability density function; (2) the underlying model is nonlinear.

A test statistic designed to detect time irreversibility is presented in Chapter 4. The statistic is the sample estimate of the symmetric-bicovariance function. Its sampling distribution is investigated in this chapter. For the independently and identically distributed case, an exact expression for the variance of the statistic is be given. It is shown that, for this case, the statistic is asymptotically distributed normal. By way of an approximate expression, I show that the variance of the statistic is much larger in the ARMA case relative to the independently and identically distributed case. This motivates the transformation, in the ARMA case, to residuals from an ARMA model fitted to the original data. A portmanteau version of the estimated symmetric-bicovariance function is also studied in this chapter. Monte Carlo results are presented to study the small sample properties of the test under the null hypothesis

In Chapter 5, I estimate the power of the test. Through Monte Carlo simulations I track the power of the test against Bilinear and Threshold Autoregressive models. Power comparisons are made against both the BDS test and Hinich's test.

I apply the tests to economic and financial time series in Chapter 6. For each series, I first calculate a set of portmanteau statistics on ARMA residuals. Then, I estimate the symmetric-bicovariance function on the ARMA residuals.

Chapter 7 concludes the dissertation. The results of the preceding sections are summarized and elaborated. Suggestions for future work are made.

2. Literature Survey

A. Empirical Work on the Business Cycle Asymmetry Hypothesis

An unresolved issue in business cycle analysis is whether the business cycle is symmetric. Claims that the business cycle is asymmetric can be traced back at least to Mitchell (1927) and Keynes (1936). From these and other writers came the proposition that the business cycle is asymmetric in the following sense: upturns are longer, but less steep, than downturns.

Burns and Mitchell (1947) attempted to quantify this asymmetry in the following sense. Having detrended the data by removing a log-linear trend, they first dated peaks and troughs for successive cycles using the National Bureau of Economic Research identification rules. For each cycle, the slope of the expansion was defined as the slope of the line connecting the trough to the peak. Analogously, the slope of the contraction was defined as the absolute value of the slope of the line connecting the peak to the trough. According to the asymmetry hypothesis, the average slope of expansions should differ from the average slope of contractions.[7] Their test of asymmetry thus

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[7] Blatt (1980) more recently proved that this equality between the average expansion and contraction slopes is a property of all Frisch-type models.

consisted of checking whether these two slopes indeed differed for several business cycle indicators.

For many financial and production series, they found evidence of asymmetry in the sense defined above. However, no strict hypothesis testing was carried out. A reasonable test would seem to be a two sample goodness of fit test to see if the set of expansion slopes was drawn from the same probability distribution function as the set of contraction slopes. But standard procedures, such as the Kolmogorov-Smirnov and Cramer-von Mises tests, would not be appropriate due to the lack of independence within and across the two sets of slope measurements.[8]

B. Neftci's Markov Chain Test for Asymmetry

Attempts in the recent literature to resolve the question of whether business cycles are asymmetric are mixed. Neftci (1984) reopened this issue with evidence suggesting that the time series behavior of several alternative definitions of the aggregate quarterly unemployment rate is asymmetric.

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[8] Blatt (1983, p. 242) ignored this lack of independence when he tested, with the Cramer-von Mises test, whether the two sets of slopes for pigiron production were drawn from the same probability distribution function.

Letting $\{X_t\}$ be a stationary economic time series, Neftci defined the state-indicator sequence $\{I_t\}$ by:

$$I_t = \begin{cases} 1 \text{ if } \Delta X_t > 0 \\ 0 \text{ if } \Delta X_t \leq 0 \end{cases} \tag{2.1}$$

He made the following two assumptions about the sequence $\{I_t\}$: (1) $\{I_t\}$ is a stationary stochastic process; and (2) $\{I_t\}$ is a second-order Markov procèss. These two assumptions were later made by Neftci and McNevin (1986) and Falk (1986).

Let $S_T = \{i_1, i_2, \ldots, i_T\}$ denote a realization of $\{I_t\}$. The log-likelihood function corresponding to a given realization S_T of $\{I_t\}$ is:

$$\begin{aligned} L(S_T, p_{111}, p_{101}, \ldots) = \; & n_{111} \cdot \log(p_{111}) + n_{011} \cdot \log(1-p_{111}) + \\ & n_{101} \cdot \log(p_{101}) + n_{001} \cdot \log(1-p_{101}) + \\ & n_{010} \cdot \log(p_{010}) + n_{110} \cdot \log(1-p_{010}) + \\ & n_{000} \cdot \log(p_{000}) + n_{100} \cdot \log(1-p_{000}) + \log(\pi_{00}), \end{aligned} \tag{2.2}$$

where

$p_{kji} = \text{Prob}(I_t = k \mid I_{t-1} = j, I_{t-2} = i)$,

n_{kji} = the number of occurrences of $(I_t = k \mid I_{t-1} = j, I_{t-2} = i)$,

$\pi_{00} = \text{Prob}(I_2 = i_2, I_1 = i_1)$,

$k,j,i = 1,2$ and $t = 3, \ldots, T$.

The test of symmetry Neftci considered is:

$$H_0: \quad p_{111} = p_{000} \tag{2.3}$$

In testing (2.3), one maximizes (2.2) twice: under the alternative hypothesis (an unconstrained maximization problem) and again subject to the constraint (2.3). The resultant (log) likelihood ratio statistic (when multiplied by -2) is asymptotically distributed Chi-square with one degree of freedom. Given the presence of the initial state probability π_0, a complicated nonlinear estimation routine is required to obtain the maximum likelihood estimates of the transition probabilities.

Neftci reported evidence against (2.3) for several post-war aggregate unemployment rate series. He used quarterly data on the overall unemployment rate, unemployment rate for insured workers and the unemployment rate fifteen weeks and over. In particular he found $\hat{p}_{000} > \hat{p}_{111}$ for all these variables, a result which could not be rejected at the 80 percent confidence level.

However, Sichel (1989) identified an error in Neftci's maximum likelihood calculations. Making corrections he showed that Neftci's second-order Markov procedure provides no evidence of asymmetry in the aggregate quarterly unemployment rate. Sichel also showed that the power of Neftci's procedure is relatively low.

Under the assumption that $\{I_t\}$ is a first-order Markov process, the log-likelihood function is:

$$L(S_T, p_{11}, p_{00}) = n_{11} \cdot \log(p_{11}) + n_{01} \cdot \log(1-p_{11}) + n_{00} \cdot \log(p_{00}) + n_{10} \cdot \log(1-p_{00}) + \log(\pi_0), \tag{2.4}$$

where

$p_{ji} = \text{Prob}(I_t = j \mid I_{t-1} = i)$,

n_{ji} = the number of occurrences of $(I_t = j \mid I_{t-1} = i)$,

$\pi_0 = \text{Prob}(I_1 = i_1)$

$j, i = 1,2$ and $t = 2, \ldots, T$.

Ignoring the initial state probabilities π_{00} and π_0, and thereby using an approximate likelihood technique, Rothman (1988) tested the order of the Markov process $\{I_t\}$ by subtracting the maximized value of (2.4) from the maximized value of (2.3). When multiplied by -2, this yields a log-likelihood ratio statistic which is asymptotically distributed as Chi-square with two degrees of freedom. By ignoring initial conditions, the calculations needed to obtain estimates of the transition probabilities are greatly simplified.

For most series considered, at the 90 percent confidence level Rothman could not reject the null hypothesis that the order of $\{I_t\}$ is one. On the basis of these results, he concluded that it is more appropriate to assume the state-indicator sequence for these series is a first-order Markov process. One useful by-product of the first-order assumption is that the number of parameters to be estimated is halved and the number of degrees of freedom is increased.

Under the first order assumption, at the 80 percent confidence level Rothman could reject the null hypothesis that $\hat{p}_{00} = \hat{p}_{11}$ for the aggregate unemployment rate. This thus became the first paper to produce correctly evidence of asymmetry in the aggregate unemployment rate in a Markov chain framework.

Rothman next tested for the presence of Neftci-type asymmetry across industrial sector unemployment rates, under the first order assumption. The goal was to isolate those sectors which are the sources of the aggregate asymmetric behavior. His main finding was that the manufacturing sector drives the aggregate unemployment rate asymmetry.

Falk [1986] applied Neftci's test to quarterly U.S. real GNP, investment and productivity data and to production indexes for five other O.E.C.D. countries.[9] He found $\hat{p}_{111} > \hat{p}_{000}$ for U.S. GNP and investment and the opposite for productivity. For the first two indicators the hypothesis that $\hat{p}_{111} > \hat{p}_{000}$ could be rejected at the 80 percent confidence level but $\hat{p}_{111} < \hat{p}_{000}$ could not be rejected for

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9 Most of these time series are nonstationary. Falk thus applied several different trend removal procedures and reported that his results are not sensitive to the detrending procedure employed.

productivity. To the extent that productivity is a pro-cyclical indicator, this is evidence of asymmetry in the direction opposite to that initially suggested by Mitchell and Keynes. For the O.E.C.D. countries, all hypotheses of asymmetry could be rejected. Falk thus concluded that the asymmetric business cycle hypothesis is the less compelling of the two.

C. DeLong and Summers' Skewness Test for Asymmetry

DeLong and Summers (1986) take a different approach in testing for business cycle asymmetry. Without proof, they claimed that the asymmetry hypothesis implies there should be skewness in the marginal frequency distribution of real GNP growth rates. Accordingly, they tested the condition that the skewness coefficient for this series is non-zero.

Since real GNP growth rates are not independently distributed, they decided to estimate the sampling variability of their skewness estimates by Monte Carlo simulation. First, an AR(3) process was estimated for the time series of growth rates. It was then used to generate 300 artificial series for the sample period under the assumption that the shocks to the autoregressive process were distributed standard normal. The empirical standard deviation of the estimated skewness under the null hypothesis was then calculated as the standard deviation of the skewness of the artificially generated series. Using this procedure they failed to reject symmetry for real GNP growth rates

There are two main problem with the DeLong and Summers approach. First, the AR(3) specification is inconsistent with identification through standard Box-Jenkins analysis. Usually an MA(1) or MA(2) specification is reported in the literature; see for example Nelson and Plosser (1982). As such, the standard errors they reported may be quite different from their true values. Second, they offered no verification for the assumption of normality for the estimated real GNP growth rate innovations. It probably would have been more appropriate to bootstrap with these estimated innovations.

Moreover, they presented no evidence on the power of their test. Welsh and Jernigan (1983) reported that estimated skewness coefficients had very low power against the asymmetric alternatives they studied. Thus, the power of DeLong and Summers' procedure may indeed be very low.

D. The BDS Statistic and Hinich's Bispectrum Test

Traditional economic time series analysis has been dominated by the class of linear Gaussian models. The test statistic introduced in this dissertation is designed to capture a property which these time series models do not possess. As such, the test is in line with two recent time series developments geared towards the detection of nonlinear and non-Gaussian behavior in economic time series: (1) the BDS statistic; and (2) the Hinich bispectrum test.

-The BDS Statistic-

Economists have recently become interested in testing for low dimensional chaos in economic and financial data. Observed time series generated by chaotic processes appear to be random utilizing conventional time series methods such as time series plots, the autocorrelation function and spectral analysis.

The correlation dimension, a measure of the relative rate of scaling of the density of points within a given space, permits a researcher to obtain topological information about the underlying system generating the observed data without requiring a prior commitment to a given structural model. If the time series is a realization of a random variable, the correlation dimension estimate should increase monotonically with the dimensionality of the space within which the points are contained. By contrast, if a low correlation dimension is obtained, this provides an indication that additional structure exists in the time series. In this way, the correlation dimension estimates may prove useful to economists wishing to scrutinize uncorrelated time series or the residuals from fitted linear time series models for information on possible nonlinear structure.

Brock, Dechert and Scheinkman (BDS) (1988) provided asymptotic results for the distribution of the standardized correlation integral when the observed points are generated by an independently and identically distributed set of random variables. The standardized correlation integral is commonly referred to as the BDS statistic.

Any sequence of points, $\{X_t\}$, $t = 1,\ldots,T$, can be transformed into a sequence of d-tuples:

$$\{(X_{t1}, X_{t2}, \ldots, X_{td})\}.$$

These d-tuples, regarded as points in a d-dimensional Euclidian space, can be "plotted" and properties of the cloud of points so created examined. The choice of the value of 'd' is the choice of "embedding dimension"; it is the size of the Euclidian space into which the original is being fitted.

If the generated points are from observations on a random variable, then as d, the embedding dimension, is increased without bound and assuming an unlimited sample size, the size of the space into which the d-tuples will fit is 'd' for all values of 'd'; that is, random variables are space filling. But if the points are generated by a mechanism that is deterministic, or at least one that produces a shape that requires only 'k' dimensions, then as the embedding dimension is increased without limit, the dimension of the points will not increase beyond 'k'.

The sample correlation integral is given by:

$$C_D(t,T) = T^{-2} \sum_{i,j} \theta(r - |X_i - X_j|),$$
$$r>0,$$
$$X_i = (X_i, X_{i+1}, \ldots X_{i+d-1}).$$

$\theta(\cdot)$ is the Heaviside step function which maps positive arguments into one, and non-positive arguments into zero. Thus, $\theta(\cdot)$ counts the number of points which are within distance 'r' from each other. 'r' is called the scaling parameter.

The BDS statistic is formed as follows:

$$w_D(r,T) = \sqrt{T}\ [\ C_D(t,T) - C_1(t,T)^D\ ] / \sigma_D(r,T), \text{ where}$$

$$\sigma_D(r,T)^2 = 4\ [\ K^D + 2 \sum_{j=1}^{j=D-1} K^{D-j} C^{2j} + (D-1)^2 C^{2D} - D^2 K C^{2D-2}\ ],$$
$$C = C(r) = \int [F(z+r) - F(z-r)]\, dF(z),$$
$$F(\cdot) = \text{cumulative distribution function for } \{X_t\},$$
$$K = K(r) = \int [F(z+r) - F(z-r)]^2\, dF(z).$$

Brock, Dechert and Scheinkman (1988) showed that $w_D(r,T)$ is asymptotically N(0,1) under the null hypothesis that $\{X_t\}$ is independently and identically distributed.

The BDS statistic is sensitive to many deviations from independence. Hsieh and LeBaron (1988) showed that the BDS statistic has good power against the null hypothesis of independence. The alternative hypothesis is very broad as it includes not only chaotic attractors, but also linear and nonlinear stochastic processes. Brock (1988) showed that the statistic is usefully applied to the residuals of estimated times series models.

In Chapter 4 below I compare the rate of convergence to the asymptotic distribution for the BDS statistic and the time irreversibility test statistic I introduce. Also, in Chapter 5 I compare the power of the two statistics for two different members of the alternative hypotheses.

-Hinich's Bispectrum Test-

Let $\{X_t\}$ be a real valued third order stationary process with mean μ. The third order central moments $C(t_1,t_2)$ is defined as:

$$C(t_1,t_2) = E[\ (X_t-\mu)\ (X_{t+t_1}-\mu)\ (X_{t+t_2}-\mu)\ ] \tag{2.5}$$

The bispectrum is the double Fourier transform of the third order cumulant function. More specifically, the bispectrum is defined for frequencies ω_1 and ω_2 in the domain:

$$\Omega = \{\ 0 < \omega_1 < .5,\ \omega_2 < \omega_1,\ 2\omega_1 + \omega_2 < 1\},\ \text{as} \tag{2.6}$$

$$B(\omega_1,\omega_2) = (1/2\pi)^{k-1} \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} C(t_1,t_2)\exp[-\iota(\omega_1 t_1+\omega_2 t_2)],$$

The skewness function $\Gamma(\omega_1,\omega_2)$ is defined in terms of the bispectrium as follows:

$$\Gamma^2(\omega_1,\omega_2) = |B(\omega_1,\omega_2)|^2 / S(\omega_1)\ S(\omega_2)\ S(\omega_1+\omega_2), \tag{2.7}$$

where $S(\omega)$ is the spectrum of $\{X_t\}$ at frequency ω.

The following two results due to Brillinger (1965) provide the basis of Hinich's (1982) bispectrum test: (1) if $\{X_t\}$ is Gaussian, $\Gamma(\omega_1,\omega_2)$ is zero over all frequencies ω_1, ω_2 in Ω; and (2) $\Gamma(\omega_1,\omega_2)$ is constant overall frequencies ω_1, ω_2 in Ω if $\{X_t\}$ is linear. Hinich produced a consistent and asymptotically complex normal estimator of the skewness function $\Gamma(\omega_1,\omega_2)$.

Call this estimator $\hat{\Gamma}(\omega_1,\omega_2)$. Hinich showed that $2|\hat{\Gamma}(\omega_1,\omega_2)|^2$ is approximately distributed as noncentral chi-squared with two degrees of freedom. Let P denote the number of frequency pairs in the principal domain Ω. Then for all i and j such that the lattic square lies entirely within the principle domain, define the test statistic:

$$\text{CHISUM} = 2\ \Sigma_i\ \Sigma_j\ |\hat{\Gamma}(\omega_i,\omega_j)|^2 \tag{2.8}$$

Under the null hypothesis that $\{X_t\}$ is Gaussian and thus $B(\omega_1,\omega_2)$ is identically zero, Hinich proved that CHISUM is approximately distributed as central chi-square with two degrees of freedom. The Hinich linearity test uses the empirical distribution of $\{2|\hat{\Gamma}(\omega_1,\omega_2)|\}$ in the principal domain to test the hypothesis that the $\hat{\Gamma}(\omega_1,\omega_2)$'s are not all the same. The 80th quantile of these statistics is a robust single-test statistic for this dispersion.

Ashley, Patterson and Hinich (1986) proved an important equivalence theorem which states that the Hinich bispectral linearity test statistic is invariant to linear filtering of the data. More specifically, if $\{Y_t\}$ is generated by passing $\{X_t\}$ through a fixed, causal, linear filter with absolutely summable impulse response weights, then $\{X_t\}$ and $\{Y_t\}$ have identical squared skewness functions. Thus, the linearity test can be either applied to the raw series or the residuals of a linear model. For the following nonlinear autoregressive model:

$$X_t = [.5 + .5\epsilon_{t-1}]\cdot X_{t-1} + \epsilon_t \tag{2.9}$$

Ashley, Patterson and Hinich showed that the Hinich linearity test is equally powerful in detecting the nonlinearity regardless of whether the source or residual series (from an AR(2) fit) is used.

The estimated size and power of Hinich's bispectrum test and the time irreversibility test are compared in Chapters 4 and 5 below.

-The BDS and Hinich Tests and Model Specification-

In concluding this section, I wish to note the following feature of both the BDS and Hinich tests. While for some members of the respective alternative hypotheses both tests have good power, neither serves as a direct guide to specification of an appropriate time series model. More specifically, both tests yield little information about the source of deviation from the respective null hypotheses.

A good example of this is the recent paper by Stokes and Hinich (1989). They first rejected linearity for the Box and Jenkins (1970) gas flow data. The modeling strategy subsequently adopted, the addition of terms to reduce the residual variance on the output series, was suggested by the Hinich test only to the extent that these alternative specifications involved the addition of nonlinear terms.

3. Time Reversibility

One motivation for my dissertation topic is to present a statistical test that can help decide whether actual business cycles are symmetric or asymmetric. I begin by explaining the direction of recent research in the analysis of two celebrated data sets in the statistical time series literature, the Canadian lynx and the Wolf sunspot data series. The reason why I first focus on these series is that, as explained below, they exhibit time series properties consistent with the asymmetric business cycle hypotheses.

The Canadian lynx data set consists of records of annual Canadian lynx trappings around the Mackenzie River from 1821 to 1934 as recorded by the Hudson Bay Company[10]. The sunspot series is comprised of measurements, dating back to 1700, of annual means of the sunspot (or Wolf's relative) number, which is given by[11]:

$R = K(10g + f)$, where

g = the number of groups of sunspots

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10 The long history of the linear time series analysis of this data set is reviewed in Campbell and Walker (1977).

11 This formula was proposed by Rudolf Wolf of Zurich in 1848. Yule (1927) introduced the class of linear autoregressive models in his famous study of this series. The large literature of linear time series analysis of the data set that followed Yule's seminal paper is reviewed in Morris (1977).

f = the total number of sunspots

K = a constant for the observatory where the observations are made

Over roughly the past ten years a consensus has developed that it is necessary to employ nonlinear time series methods in order to model these series appropriately; with respect to the Canadian lynx data, see Campbell and Walker (1977) and Tong (1977); and for the sunspot numbers series, see Tong and Lim (1980), Ghaddar (1980) and Lim (1981). A prominent property of both these series which has led to this conclusion is that, while both are cyclical, there appears to be an asymmetry between the lengths of ascent and descent periods. That is, in each cycle the gradient of the rise to the maximum differs from the gradient of the fall to the next minimum[12]. This feature has led both Tong (1983) and Subba Rao and Gabr (1984) to claim that these two series are time irreversible.

A general discussion of time reversibility is found in Tong (1983, pp. 25-31). A major theme of Tong's is that linear Gaussian ARMA models are not applicable to data exhibiting time irreversibility.

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12 See Tong (1983, pp. 166 and 231) for a tabulation of the asymmetry between the lengths of ascensions and descensions in these two series.

A. Definition of Time Reversibility and Some Time Reversible Processes

The formal statistical definition of time reversibility is:

Definition 1: A time series $\{X_t\}$ is time reversible if for every positive integer n, and every $t_1, t_2, \ldots, t_n \in \mathbf{R}$, the vectors $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ and $(X_{-t_1}, X_{-t_2}, \ldots, X_{-t_n})$ have the same joint probability distributions. A time series which is not time reversible is said to be time irreversible.

Note that the above definition does not impose stationarity on the time series $\{X_t\}$. This is in contrast to an alternative definition of time reversibility found, for example, in Tong (1983) and Subba Rao and Gabr (1984), that requires stationarity.

I shall next show three cases for which $\{X_t\}$ is time reversible: (1) $\{X_t\}$ is independently and identically distributed; (2) $\{X_t\}$ is independently, but not identically, distributed; (3) $\{X_t\}$ is a stationary Gaussian process, but not necessarily independent.

Lemma 3.1: Let $\{X_t\}$ be a stationary process consisting of a sequence of independently and identically distributed random variables, then $\{X_t\}$ is time reversible.

Proof: By the independence assumption, the joint probability distribution functions can be re-expressed as:

$$F_{t_1,\ldots,t_n}(x_{t_1},\ldots,x_{t_n}) = F_{t_1}(x_{t_1})\cdots F_{t_n}(x_{t_n})$$
and (3.1)
$$F_{-t_1,\ldots,-t_n}(x_{-t_1},\ldots,x_{-t_n}) = F_{-t_1}(x_{-t_1})\cdots F_{-t_n}(x_{-t_n})$$

Since $\{X_t\}$ is identically distributed:

$$F_t(x_t) = F_{t'}(x_{t'}),\ t \neq t',\ \forall\ t \text{ and } t' \qquad (3.2)$$

By (3.1) and (3.2) it is seen that $F_{t_1,\ldots,t_n}(x_{t_1},\ldots,x_{t_n}) = F_{-t_1,\ldots,-t_n}(x_{-t_1},\ldots,x_{-t_n})$, so that $\{X_t\}$ is time reversible.

Example 3.1: Let $\{X_t\}$ be the process defined by the sequence of independently, but not identically, distributed random variables where $F_t(x_t) = N(\mu\cdot t^2,\sigma^2)$, then $\{X_t\}$ is time reversible and clearly non-stationary.

By the independence assumption, the joint probability distribution functions can be re-expressed as:

$$F_{t_1,\ldots,t_n}(x_{t_1},\ldots,x_{t_n}) = F_{t_1}(x_{t_1})\cdots F_{t_n}(x_{t_n})$$
and (3.3)
$$F_{-t_1,\ldots,-t_n}(x_{-t_1},\ldots,x_{-t_n}) = F_{-t_1}(x_{-t_1})\cdots F_{-t_n}(x_{-t_n})$$

Since $F_t(x_t) = N(\mu\cdot t^2,\sigma^2)$:

$$F_t(x_t) = F_{-t}(x_{-t}),\ \forall\ t, \qquad (3.4)$$

then by (3.3) and (3.4) it is seen that $F_{t_1,\ldots,t_n}(x_{t_1},\ldots,x_{t_n}) = F_{-t_1,\ldots,-t_n}(x_{-t_1},\ldots,x_{-t_n})$, so that $\{X_t\}$ is time reversible.

The importance of this example is that there exists a non-stationary process that is time reversible, so that non-stationarity does not imply time irreversibility. As is well known, see Tong (1983) and Subba Rao and Gabr (1984), stationarity does not imply time reversibility. Hence, stationarity and time reversibility are separate concepts and neither implies the other.

Below I shall assume, unless otherwise indicated, that $\{X_t\}$ is stationary. I do this because only in the case of stationarity have I developed a fairly complete theory of time reversibility and of the distribution of the relevant test statistics.

The assumption of stationarity simplifies the analysis of reversibility and yields simpler, but more restrictive definitions of time reversibility. Suppose $\{X_t\}$ is time reversible. By the assumed stationarity of $\{X_t\}$, $(X_{-t_1}, X_{-t_2}, \ldots, X_{-t_n})$ and $(X_{-t_1+m}, X_{-t_2+m}, \ldots, X_{-t_n+m})$ have the same joint distributions for any integer m. Consider the special case in which the time indices $\{t_i\}$ are constructed as follows: $t_i = t_{i-1} + k$, $k \in \mathbb{R}$, $i = 2,\ldots,n$ i.e., the set $\{t_i\}$ is characterized by equal, not necessarily integer, increments of time. Letting $m = t_1 + t_n$, it is seen that for a stationary time series $\{X_t\}$, time reversibility implies that the vectors $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ and $(X_{t_n}, X_{t_{n-1}}, \ldots, X_{t_1})$ have the same joint probability distributions; this is a stationary restricted definition.

Without imposing stationarity in the definition of time reversibility, Lawrance (1988) restricted the elements of the sequence of time indices, $\{t_i\}$, to be separated by equal increments. Under this restriction he showed that time reversibility implies stationarity.

Lemma 3.2: Let $\{X_t\}$ be a stationary Gaussian process, then $\{X_t\}$ is time reversible.

Proof: If $\{X_t\}$ is a stationary Gaussian process with null mean vector, but not necessarily diagonal covariance matrix, then the joint probability density function of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is (see Priestley (1984, p. 90)):

$$f_{t_1,\ldots,t_n}(x_1,\ldots,x_n) = k^{-1} \exp[-(1/2)z],$$

where:

$$k = (2\pi)^{n/2}\Delta^{1/2}$$

$$\Delta = |\Sigma| = \det(\Sigma),\ \Sigma = \{\sigma_{ij}\},\ \Sigma^{-1} = \{\sigma^{ij}\} \qquad (3.5)$$

$$\Sigma = \text{covariance matrix of } (X_{t_1}, X_{t_2}, \ldots, X_{t_n})$$

$$z = \sum_{i=t_1}^{t_n} \sum_{j=t_1}^{t_n} \sigma^{ij}(x_i \cdot x_j)$$

But Δ and z are invariant to reversal of the order of the indices (i,j). Hence, $\{X_t\}$ is time reversible.

Note that from Lemma 3.2 it follows that all Gaussian ARMA models are time reversible. As such, an analyst who wishes to model a particular time series as a Gaussian ARMA process should indeed confirm that the observed data are time reversible. More specifically, if time irreversibility is discovered in the given time series a Gaussian ARMA approach would be an incorrect one to adopt. The test presented below is designed to detect such time irreversibility.

The result that stationary Gaussian processes are time reversible appeared as Theorem 1 in Weiss (1975, p. 831). In the same paper, Weiss proved the converse of this result within the context of discrete-time ARMA models; this theorem was the main contribution of his paper. According to Weiss's Theorem 2, if $\{X_t\}$ is a stationary time reversible autoregressive moving average process given by:

$$X_t = \sum_{k=1}^{p} \alpha_k X_{t-k} + \sum_{\ell=1}^{q} \theta_\ell \epsilon_{t-\ell}, \qquad (3.6)$$

with $\{\epsilon_t\}$ an independent and identically distributed sequence of non-degenerate random variables,

then the underlying sequence of independently and identically distributed innovations, $\{\epsilon_t\}$, are normally distributed. This result strictly holds only if $p \neq 0$, or if $p = 0$, the cases $\theta_\ell = \theta_{M-\ell}$ (ℓ =

$0,1,\ldots,M)$, $\theta_\ell = -\theta_{M-\ell}$ ($\ell = 0,1,\ldots,M$) are excluded. The exclusion is necessary since if $\theta_\ell = \theta_{M-\ell}$ for $\ell = 0,\ldots,M$, $\{X_t\}$ is time reversible irrespective of the distribution of $\{\epsilon_t\}$, and if $\theta_\ell = -\theta_{M-\ell}$ for $\ell = 0,\ldots,M$, then $\{X_t\}$ is time reversible whenever $\{\epsilon_t\}$ has a symmetric probability distribution. See Lemma 1 of Weiss's article.

Weiss conjectured, without proof, that this result holds when $\{X_t\}$ is a general linear process. This conjecture was shown to be true in a recent paper by Hallin, Lefevre and Puri (1988).

B. A Property of Stationary Time Reversible Processes

I next establish the equality between certain pairs of moments from the joint probability distributions for a time reversible stationary time series $\{X_t\}$.

Lemma 3.3: Let $\{X_t\}$ be a stationary time series and assume that the multivariate characteristic generating functions of (X_t,X_{t-k}) and (X_{t-k},X_t) can be expanded as a convergent series in the moments and cross moments of the respective joint probability distributions; that is, it is assumed that the joint probability distributions are uniquely characterized by the respective sequence of moments and cross moments (see Kendall and Stuart (1962), Vol. I, pp. 109-110).

If $\{X_t\}$ is time reversible, then:

$$E[X_t^i \cdot X_{t-k}^j] = E[X_t^j \cdot X_{t-k}^i], \tag{3.7}$$

for all i,j,k ∈ **N**, where the expectation is taken with respect to each respective joint distribution.

But, if $\{X_t\}$ is time irreversible in the sense that $F_{t,t-k}(x_t, x_{t-k}) \neq F_{t-k,t}(x_{t-k}, x_t)$, then:

$$E[X_t^i \cdot X_{t-k}^j] \neq E[X_t^j \cdot X_{t-k}^i], \tag{3.8}$$

for some i,j,k ∈ **N**.

<u>Proof</u>: By the definition of mathematical expectation:

$$E[X_t^i \cdot X_{t-k}^j] = \int_{X_t}\int_{X_{t-k}} X_t^i \cdot X_{t-k}^j dF_{t,t-k}(\cdot) \tag{3.9}$$

and

$$E[X_{t-k}^i \cdot X_t^j] = \int_{X_{t-k}}\int_{X_t} X_{t-k}^i X_t^j \, dF_{t-k,t}(\cdot) \tag{3.10}$$

If $\{X_t\}$ is time reversible, then $F_{t,t-k}(\cdot) = F_{t-k,t}(\cdot)$. Thus, equation (3.9) equals equation (3.10) for all i,j,k ∈ **N** and condition (3.7) holds. Likewise, if $F_{t,t-k}(x_t, x_{t-k}) \neq F_{t-k,t}(x_{t-k}, x_t)$ for some k, then equation (3.9) does not equal equation (3.10) for some i,j,k ∈ **N** and statement (3.8) is true. Equation (3.9) does not equal equation (3.10) for if not, the assumed uniqueness of the representation of the distributions by the moments would be violated.

For $i = j = 1$:

$$E[X_t \cdot X_{t-k}] = E[X_t \cdot X_{t-k}] \tag{3.11}$$

for all positive integers k.

Statement (3.11) is simply the tautology that the autocovariance of a stationary time series at lag k is equal to itself. This is because the autocorrelation function is an even function of k. As such, it is seen that the autocovariance function can provide no relevant information with respect to the potential time irreversibility of any specific time series.

When at least one of i, j is greater than one, $i,j \in \mathbf{N}$, the two terms in (3.7) are called generalized autocovariances, following the terminology of Welsh and Jernigan (1983). From Lemma 3.3 it follows that if there exists a lag k for which these two moments do not equal one another, the series is time irreversible. While this is a sufficient condition for time irreversibility, it is not a necessary one, since (3.7) considers only a proper subset of moments from the joint distributions of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ and $(X_{t_n}, X_{t_{n-1}}, \ldots, X_{t_1})$; that is, above I consider only the pairs (X_t, X_{t-k}) and (X_{t-k}, X_t)[13].

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[13] I do not consider the case, for example, where $F_{t_1,t_2}(\cdot) = F_{t_2,t_1}(\cdot)$ but $F_{t_1,t_2,t_3}(\cdot) \neq F_{t_3,t_2,t_1}(\cdot)$ for some $t_1, t_2, t_3 \in \mathbf{Z}$, a situation that is unlikely to occur in practice.

C. The Symmetric-Bicovariance Function

I propose to consider the difference between two bicovariances. I define the symmetric-bicovariance functions:

$$\gamma_{2,1}(k) = \{E[X_t^2 \cdot X_{t-k}] - E[X_t \cdot X_{t-k}^2]\}$$
$$\text{and} \qquad (3.12)$$
$$\gamma_{1,2}(k) = \{E[X_t \cdot X_{t-k}^2] - E[X_t^2 \cdot X_{t-k}]\}$$

for all integer values of k.

Note that $\gamma_{2,1}(k) = -\gamma_{1,2}(k) \ \forall \ k \in \mathbf{N}$. If $\{X_t\}$ is time reversible, then $\gamma_{2,1}(k) = \gamma_{1,2}(k) = 0 \ \forall \ k \in \mathbf{N}$. My reason for looking only at the differences in bicovariances is that the distributional properties will be more manageable than for the higher-order moments and that as a practical matter the lower-order moments seem to be sufficiently informative[14].

I shall demonstrate the time irreversibility of two different time series models using the symmetric-bicovariance function, one linear and one nonlinear. First, consider the following non-Gaussian MA(1) model:

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[14] I am indebted to Pomeau (1982) for the suggestion of studying time reversibility through a similar but higher-order function, $\gamma_{3,1}(k)$. He did not, however, draw a direction connection between his proposed test and the formal statistical definition of reversibility. He also did not investigate the sampling of any estimator of $\gamma_{3,1}(k)$.

$$X_t = \epsilon_t - \theta\epsilon_{t-1}, \tag{3.13}$$

where $\{\epsilon_t\}$ is a sequence of independent and identically distributed random variables drawn from a non-symmetric probability distribution function; $\theta \neq -1$.

It is straightforward to show that:

$$E[X_t^2 \cdot X_{t-1}] = E[(\epsilon_t^2 - 2\theta\epsilon_t\epsilon_{t-1} + \theta^2\epsilon_{t-1}^2)\cdot(\epsilon_{t-1} - \theta\epsilon_{t-2})]$$
$$= \theta^2\mu_3^\epsilon, \text{ where } \mu_3^\epsilon = E[\epsilon_t^3]$$

and that (3.14)

$$E[X_t \cdot X_{t-1}^2] = E[(\epsilon_t - \theta\epsilon_{t-1})\cdot(\epsilon_{t-1}^2 - 2\theta\epsilon_{t-1}\epsilon_{t-2} + \theta^2\epsilon_{t-2}^2)]$$
$$= -\theta\mu_3^\epsilon$$

Because $\gamma_{2,1}(1) = (\theta^2 + \theta)\mu_3^\epsilon$ is non-zero under the given assumptions, we conclude that $\{X_t\}$ is time irreversible. Note that if $\theta = -1$, $X_t = \epsilon_t + \epsilon_{t-1}$, which is obviously time reversible.

Next, consider the following bilinear model:

$$X_t = \alpha X_{t-1} + \beta X_{t-1}\epsilon_{t-1} + \epsilon_t, \tag{3.15}$$

where $\{\epsilon_t\}$ is a sequence of independent and identically distributed N(0,1) random variables.

It can be shown (see Subba Rao and Gabr (1984, pp. 53-57)) that:

$$E[X_t^2 \cdot X_{t-1}] = \alpha^2\mu_3 + \beta^2 Q_2 + 2\alpha\beta Q_1 + \mu,$$

and

$$E[X_t \cdot X_{t-1}^2] = \alpha\mu_3 + \beta Q_1, \tag{3.16}$$

where

$$\mu = E[X_t] = \beta / (1 - \alpha),$$

$$\mu_2 = E[X_t^2] = (1 + 2\beta^2 + 4\alpha\beta\mu) / (1 - \alpha^2 - \beta^2),$$

$$\mu_3 = E[X_t^3] = (1 - 2\alpha\beta^2 - \alpha^3)^{-1} \cdot \{\beta^3 Q_3 + 3\alpha^2\beta Q_1 + 3\mu(1 + 6\alpha\beta^2)\},$$

$$Q_1 = E[X_{t-1}^3 \cdot \epsilon_{t-1}] = (3 / 1 - \beta^2) \cdot (1 + \alpha^2\mu_2 + 2\beta^2 + 4\alpha\beta\mu)$$

$$Q_2 = E[X_{t-1}^3 \cdot \epsilon_{t-1}^2] = (1 - 3\alpha\beta^2)^{-1} \cdot (\alpha^3\mu_2 + \beta^3 Q_3 + 3\alpha 2\beta Q_1 + 9\mu)$$

$$Q_3 = E[X_{t-1}^3 \cdot \epsilon^3{}_{t-1}] = (3 / 1 - \beta^2) \cdot (5 + 3\beta^2 + 3\alpha^2\mu_2 + 12\alpha\beta\mu)$$

Hence, for $\{X_t\}$ given by (3.15) $\gamma_{2,1}(1) \neq 0$, showing that $\{X_t\}$ is time irreversible, except for isolated pairs of values for (α,β) that solve the above equations in (3.16) simultaneously.

The two examples demonstrate that time irreversibility as indicated by the symmetric-bicovariance function can stem from two sources: (1) the underlying model may be nonlinear even though the innovations are symmetrically (perhaps normally) distributed; or (2) the underlying innovations may be drawn from an asymmetric probability distribution while the model is linear. I shall call the first Type 1 time irreversibility and the second Type 2 time irreversibility. Type 2 time irreversibility is removable by linear transformation; that is, the residuals obtained from the inverse linear transformation are independently and identically distributed so that by Lemma 3.1 the transformed series is time reversible.

I close this section by noting the operational definition of time reversibility in the frequency domain. In particular, the k-th order cumulant spectrum for time reversible processes is real valued (see Brillinger and Rosenblatt (1967, p. 210)). A test of time reversibility in the frequency domain, then, consists of checking whether the imaginary parts of the Fourier transforms of all cumulants are zero.

4. A Test Statistic

A. The Estimated Symmetric-Bicovariance Function

The test statistic I propose to employ to check for the presence of time irreversibility consists of a sample estimate of the symmetric-bicovariance function given by equation (3.12). The sample bicovariances for a stationary time series $\{X_t\}$ with T observations are:

$$\hat{B}_{2,1}(k) = (T-k)^{-1} \cdot \sum_{t=k+1}^{t=T} X_t^2 \cdot X_{t-k}$$

$$\text{and} \qquad (4.1)$$

$$\hat{B}_{1,2}(k) = (T-k)^{-1} \cdot \sum_{t=k+1}^{t=T} X_t \cdot X_{t-k}^2$$

for various integer values of k.

It is straightforward to see that $\hat{B}_{2,1}(k)$ and $\hat{B}_{1,2}(k)$ are unbiased estimators of the bicovariances $B_{2,1}(k) = E[X_t^2 \cdot X_{t-k}]$ and $B_{1,2}(k) = E[X_t \cdot X_{t-k}^2]$, respectively.

I turn next to the consistency of the bicovariance estimators. In addition to the restrictions on $\{X_t\}$ made above, I now require that the sequence $\{X_t\}$ have finite sixth-order moments and assume all second and third order moments to be in ℓ_1. By Theorem 1 of

Rosenblatt and Van Ness (1965, p. 1125), the variance of these estimators goes to zero as $T\to\infty$. This result, along with the asymptotic unbiasedness of the estimators (which follows from their small sample unbiasedness), establishes that $\hat{B}_{2,1}(k)$ and $\hat{B}_{1,2}(k)$ converge in quadratic mean to $B_{2,1}(k)$ and $B_{1,2}(k)$ and that they are consistent.

With the bicovariance estimates from (4.1), the test statistic is constructed as follows:

$$\hat{\gamma}_{2,1}(k) = \hat{B}_{2,1}(k) - \hat{B}_{1,2}(k) \tag{4.2}$$

for various integer values of k. $\hat{\gamma}_{2,1}(k)$, as a linear function of $\hat{B}_{2,1}(k)$ and $\hat{B}_{1,2}(k)$, is unbiased and consistent and converges in quadratic mean to $\gamma_{2,1}(k)$.

Formally, through $\hat{\gamma}_{2,1}(k)$ I test only for time reversibility as exhibited in the bicovariance function. If one were interested in testing for time reversibility as exhibited by higher-order moments, the test statistics in (4.2) can easily be generalized to arbitrary $\gamma_{i,j}(k)$, $i,j \geq 2$.. In this case, though, caution should be exercised in that the estimates of very high order moments have relatively high standard errors.

B. The Sampling Distribution of $\hat{\gamma}_{2,1}(k)$ in the I.I.D. Case

Under the null hypothesis that $\{X_t\}$ is time reversible, the expected value of $\gamma_{2,1}(k)$ is zero for all k. In order to make the test operational, though, it is necessary to examine the sampling distribution of the test statistic.

By definition, the variance of $\hat{\gamma}_{2,1}(k)$ is:

$$\begin{aligned} Var(\hat{\gamma}_{2,1}(k)) = Var(\hat{B}_{2,1}(k)) + Var(\hat{B}_{1,2}(k)) \\ - 2\cdot Cov(\hat{B}_{2,1}(k),\hat{B}_{1,2}(k)) \end{aligned} \qquad (4.3)$$

I begin by deriving the variance for $\hat{\gamma}_{2,1}(k)$ when $\{X_t\}$ is a sequence of independently and identically distributed random variables. The exact small sample expressions for the sample bicovariances under the independently and identically distributed assumption are given in Lemma 4.1.

Lemma 4.1: Let $\{X_t\}$ be a stationary sequence of independently and identically distributed random variables for which $E[X_t] = 0 \ \forall\ t$, let μ_2 to μ_4 be defined and finite. Then:

$$\begin{aligned} &Var(\hat{B}_{2,1}(k)) = Var(\hat{B}_{1,2}(k)) = \mu_4\mu_2/(T-k) \\ &\qquad \text{and} \\ &Cov(\hat{B}_{2,1}(k),\hat{B}_{1,2}(k)) = \mu_3^2/(T-k) + \mu_2^3(T-2k)/(T-k)^2, \end{aligned} \qquad (4.4)$$

where $\mu_2 = E[X_t^2]$

$\mu_3 = E[X_t^3]$

$\mu_4 = E[X_t^4]$

<u>Proof</u>: By the independently and identically distributed assumption and since $E[X_t] = 0$, $E[X_t^2 \cdot X_{t-k}] = 0$, $\forall\ k \in \mathbf{N}$. Thus, $Var(\hat{B}_{2,1}(k)) = E[\hat{B}_{2,1}(k)^2]$. That is:

$$Var(\hat{B}_{2,1}(k)) = E[(T-k)^{-2} \cdot \sum_{t=k+1}^{t=T} \sum_{s=k+1}^{s=T} X_t^2 \cdot X_{t-k} \cdot X_s^2 \cdot X_{s-k}]. \tag{4.5}$$

Since:

$$E[X_t^2 \cdot X_{t-k} \cdot X_s^2 \cdot X_{s-k}] = \mu_4 \mu_2, \text{ for } t = s$$
$$\text{and} \tag{4.6}$$
$$E[X_t^2 \cdot X_{t-k} \cdot X_s^2 \cdot X_{s-k}] = 0, \text{ for } t \neq s,$$

and since the condition $t = s$ occurs T-k times in the calculation of $Var(\hat{B}_{2,1}(k))$, it follows from (4.5) and (4.6) that:

$$Var(\hat{B}_{2,1}(k)) = \mu_4 \mu_2/(T-k) \tag{4.7}$$

An identical argument shows that:

$$Var(\hat{B}_{1,2}(k)) = \mu_4 \mu_2/(T-k) \tag{4.8}$$

I next evaluate $Cov(\hat{B}_{2,1}(k), \hat{B}_{1,2}(k))$:

$$\mathrm{Cov}(\hat{B}_{2,1}(k),\hat{B}_{1,2}(k)) = E[(T-k)^{-2} \cdot \sum_{t=k+1}^{t=T} \sum_{s=k+1}^{s=T} X_t^2 \cdot X_{t-k} \cdot X_s \cdot X_{s-k}^2]. \qquad (4.9)$$

Since:

$$E[X_t^2 \cdot X_{t-k} \cdot X_s \cdot X_{s-k}^2] = \mu_3^2, \text{ for } s = t$$

$$E[X_t^2 \cdot X_{t-k} \cdot X_s \cdot X_{s-k}^2] = \mu_2^3, \text{ for } s = t-k \qquad (4.10)$$

and

$$E[X_t^2 \cdot X_{t-k} \cdot X_s \cdot X_{s-k}^2] = 0, \text{ for } s \neq t \text{ and } s \neq t-k,$$

and since the condition $s = t-k$ occurs $T-2k$ times in the calculation of $\mathrm{Cov}(\hat{B}_{2,1}(k),\hat{B}_{1,2}(k))$, it follows from (4.9) and (4.10) that:

$$\mathrm{Cov}(\hat{B}_{2,1}(k),\hat{B}_{1,2}(k)) = \mu_3^2/(T-k) + \mu_2^3(T-2k)/(T-k)^2 \qquad (4.11)$$

<u>Lemma 4.2</u>: Let $\{X_t\}$ be a stationary sequence of independently and identically distributed random variables for which $E[X_t] = 0 \ \forall\ t$. Then:

$$\mathrm{Var}(\hat{\gamma}_{2,1}(k)) = 2(\mu_4\mu_2 - \mu_3)/(T-k) - 2\mu_2^3(T-2k)/(T-k)^2$$

$$(4.12)$$

$$\doteq (2/T)[\mu_4\mu_2 - \mu_3 - \mu_2^3], \text{ for large } T \text{ and small } k$$

<u>Proof</u>: By (4.3) and <u>Lemma 4.1</u>.

From equation (4.12) it is seen that, for $\{X_t\}$ independently and identically distributed, $\mathrm{Var}(\sqrt{T}\hat{\gamma}_{2,1}(k)) = 2(\mu_4\mu_2 - \mu_3 - \mu_2^3)$ for large T.

If the underlying distribution is normal, $Var(\sqrt{T}\hat{\gamma}_{2,1}(k)) = 4\mu_2^3$ for large T, because $\mu_4 = 3\mu_2^2$ and $\mu_3 = 0$.

By Theorem 4.3 of Welsh and Jernigan (1983, p. 391) the bicovariance estimators, $\hat{B}_{i,j}(k)$, are asymptotically distributed as $N(0, Var(\hat{B}_{i,j}(k)))$ under the condition that $\{X_t\}$ is an independently and identically distributed sequence with finite sixth-order moments.

C. Finite Sample Properties of $\hat{\gamma}_{2,1}(k)$ in the I.I.D. Case

I examined how quickly this convergence in distribution to the normal takes place for the case in which $\{X_t\}$ is itself drawn from a normal distribution from Monte Carlo simulations using 1000 iterations. The results of these simulations appear in Table 1 which reports the kurtosis measurements, the Kolmogorov-Smirnov statistics for goodness of fit to the normal distribution and the estimated variances of $\hat{\gamma}_{2,1}(1)$ for each run.

According to the observed Kolmogorov-Smirnov D statistics, one can not reject the null hypothesis that the underlying distribution of the set of $\hat{\gamma}_{2,1}(1)$ is normally distributed for sample sizes greater than or equal to 100. This is true even at the 20% significance level except for sample size 50. The actual distributions are leptokurtotic, declining to the value for the normal.

Tables 2 through 9 contain further Monte Carlo results on the finite sampling properties of $\hat{\gamma}_{2,1}(k)$ in the independently and identically distributed case. For four different distributions, the estimated size of $\hat{\gamma}_{2,1}(k)$ was calculated and the probability of rejecting k or more times, k = 1,2,...,10, was estimated. The four distributions studied were: (1) the standard normal; (2) the chisquare with 1 degree of freedom; (3) the chisquare with 5 degrees of freedom; and (4) the standard exponential. Tables 2 through 5 contain the estimated size of $\hat{\gamma}_{2,1}(k)$ at various values of k for each case. Tables 6 through 9 contain simulation results on the probability of rejecting k or more times, k = 1,2,...,10, for each distribution.

The results on the estimated size of $\hat{\gamma}_{2,1}(k)$ strongly suggest that the convergence to the normal takes place by sample size 100 for all distributions considered. Apparently the rate of convergence to the normal is not sensitive to asymmetry of the probability density function. In Tables 6 through 9 the estimated probabilities of rejecting k or more times, k = 1,2,...,10, are compared with the probability of k or more successes in a sequence of 10 Bernoulli trials for which the probability of a success is 0.05, the size of the test under the null hypothesis. For each distribution, the estimated probabilities are close to the theoretical probabilities. This is evidence consistent with the $\hat{\gamma}_{2,1}(k)$ values being uncorrelated across

k. This result is utilized in developing a portmanteau version of the test statistics.

D. Variance of $\hat{\gamma}_{2,1}(k)$ in the ARMA Case

The sampling distribution of $\hat{\gamma}_{2,1}(k)$ when $\{X_t\}$ is independent and identically distributed provides a reference for the finite ARMA case. The exact small sample expression for $Var(\hat{\gamma}_{2,1}(k))$ when $\{X_t\}$ is ARMA is algebraically complicated and its computation is tedious. Given that exact expressions for a far less complicated set of statistics, the sample autocorrelation function, for example, are not generally known[15], the difficulty is not surprising.

Using the symbolic logic program MATHEMATICA (see Wolfram (1988)), I obtained an approximate expression for $Var(\hat{\gamma}_{2,1}(k))$ for the MA(q) case in which the underlying innovations are drawn from a symmetric probability distribution function. Because any ARMA model can be represented as an MA(∞) model, this approach provides some insight into the approximation for $Var(\hat{\gamma}_{2,1}(k))$ in the ARMA case as well.

Let $\{X_t\}$ be an invertible moving average process of order q:

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15 See Priestly (1984, pp. 330-340).

$X_t = \epsilon_t - \theta_1\epsilon_{t-1} - \theta_2\epsilon_{t-2} - \dots - \theta_q\epsilon_{t-q}$, where

$\{\epsilon_t\}$ is a sequence of random variables drawn from a symmetric p.d.f. and for which:

$E[\epsilon_t] = 0$, $E[\epsilon_t^2] = \mu_2$, and $E[\epsilon_t^4] = \mu_4$

Then, for large T, $k \geq 2q + 1$, and ignoring terms in products of the θ_i's for which the sum of the powers equals or exceeds three, an approximate expression for the variance of $\hat{\gamma}_{2,1}(k)$ is given by:

$$(2/(T-k)) \; (\; \mu_2\mu_4[1 + \sum_{i=1}^{i=q}\theta_i^2] + \mu_2^3[-1 + 3\cdot\sum_{i=1}^{i=q}\theta_i^2]) \qquad (4.13)$$

Note that if $\theta_1 = \theta_2 = \dots = \theta_q = 0$, (4.13) reduces, for large T and small k, to the result for the independently and identically distributed case given by equation (4.12). This approximation gives a lower bound on the true variance of $\hat{\gamma}_{2,1}(k)$ in the MA(q) case.

The accuracy of this approximation clearly depends upon the values of the θ_i's. Higher powers of the θ_i's were deleted in equation (4.13) on the assumption that such products are "small" relative to the main effects retained in the expression. In the appendix I give an exact expression for $Var(\hat{\gamma}_{2,1}(k))$ in the MA(2) case for $k \geq 5$.

From equation (4.13) it is seen that the variance of $\hat{\gamma}_{2,1}(k)$ for the independently and identically distributed case is less than the variance of $\hat{\gamma}_{2,1}(k)$ for the MA(q) and general ARMA(p,q) cases. This comes from recognizing that the variance of $\hat{\gamma}_{2,1}(k)$ increases as the order of the MA process increases, since as q increases, only positive terms to the variance approximation are added, for large enough k.

E. A Transformation to Reduce Variance of $\hat{\gamma}_{2,1}(k)$ in ARMA Case

It is seen then that $\text{Var}(\hat{\gamma}_{2,1}(k))$ in the ARMA(p,q) case can be large; this is especially true for nearly non-stationary time series. However, a simple transformation enables one to reduce the variance substantially, at least asymptotically. The procedure is to fit an ARMA model to the original time series, and then estimate the symmetric-bicovariance function using the nearly uncorrelated residuals. The sampling distribution for the independently and identically distributed case can then be applied, as a useful approximation for large T, which is justified by the consistency of the estimates of the model's parameters. Approximate 95% confidence intervals can be formed by taking twice the $\text{Var}(\hat{\gamma}_{2,1}(k))^{1/2}$, using estimated third and fourth moments, in accordance with the expression in equation (4.12).

Monte Carlo results on applying this procedure to several AR(1) models are presented in Tables 10 through 24. For every iteration of the Monte Carlo runs, $\hat{\gamma}_{2,1}(k)$ was calculated on the residuals from an

AR(1) model fitted to the original series. In Tables 10 through 18, results for the Gaussian AR(1) case are presented as the AR(1) coefficient varies from 0.9 to 0.1. Tables 19 through 24 present results for the AR(1) case with AR(1) coefficient equal to 0.9 and for which the innovation sequences are distributed chisquare with 1 degree of freedom, chisquare with 5 degrees of freedom and standard exponential.

For the Gaussian cases reported in Tables 10 through 18, at sample size 100 $\hat{\gamma}_{2,1}(k)$ appears to reject about twice as often as it should under the null hypothesis. The average size across k is .11 while the size under the null is .05. the estimated size improves by sample size 250. For this sample size $\hat{\gamma}_{2,1}(k)$ rejects roughly 6% of the time, slightly more often than should be rejected. By sample size 500 $\hat{\gamma}_{2,1}(k)$ appears to have converged to its asymptotic normal distribution. Note that these results are independent of the value of the AR(1) coefficient; that is the rate of convergence to the normal is the same as the AR(1) coefficient varies from 0.9 to 0.1. Thus, while the transformation to AR(1) residuals worsens the small sample properties of $\hat{\gamma}_{2,1}(k)$ relative to the independently and identically distributed case, by sample size 250 the rejection rate differs only slightly from what it should be under the null.

Convergence of $\hat{\gamma}_{2,1}(k)$ to the normal is slower for the non-Gaussian AR(1) cases reported in Tables 19 to 24. With $\chi^2(1)$

distributed innovations, at sample size 100 $\hat{\gamma}_{2,1}(k)$ rejects about two and half times more often than should be the case under the null hypothesis. The estimated size of $\hat{\gamma}_{2,1}(k)$ is reduced by a little less than one half by sample size 250. At sample size 500, the average size across k is .06 while the size under the null it is .05. The probability of rejection is reduced a bit by sample size 1000 and convergence to the normal appears to take place by sample size 5000. Matters are slightly better with $\chi^2(5)$ distributed innovations and standard exponentially distributed innovations.

F. Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ Compared to BDS and Hinich Tests

Next I compare the estimated size of $\hat{\gamma}_{2,1}(k)$ to the estimated size of the BDS and Hinich linearity test statistics. Monte Carlo results reported by Hsieh and LeBaron (1988) on the estimated size of the BDS statistic for three cases, independently and identically distributed normal, independently and identically distributed $\chi^2(4)$ and Gaussian AR(1) residuals, are presented in Tables 25 through 27. Monte Carlo results reported by Ashley, Patterson and Hinich (1986) on the estimated size of the Hinich linearity test for the independently and identically distributed normal case are reported in Table 28.

Recall that for all independently and identically distributed sequences considered, $\hat{\gamma}_{2,1}(k)$ converges to the normal distribution by sample size 100. At this sample size the BDS statistic, at embedding dimension 2, rejects from two to eight times more often than it should for the standard normal and $\chi^2(4)$ cases. Great caution then should be exercised in interpreting results at this sample size in so far as the true α-levels are far greater than the nominal values. I return to this point when comparing the estimated power of the BDS statistic against $\hat{\gamma}_{2,1}(k)$ for a threshold autoregressive alternative. For the Gaussian AR(1) residuals, at embedding dimension 2 the BDS statistic rejects from two to six times as often as it should at sample size 100. At this sample size, $\hat{\gamma}_{2,1}(k)$ uniformly rejects about twice as often as it should.

As sample size is increased to 500, the performance of the BDS statistic improves. At embedding dimension 2 and higer values of the scaling parameter r, for $\chi^2(4)$ size is about .06. For the lowest value of r reported, the BDS statstic rejects three times more often than it should. For the standard normal case matters are slightly worse, especially for smaller values of r. For the Gaussian AR(1) residuals, the estimated size of the BDS statistic is roughly .07 for higher values of r at embedding dimension 2. At this sample size, $\hat{\gamma}_{2,1}(k)$ has already converged to the normal distribution.

At sample size 1000, the estimated size of the BDS statistic is still higher than it should be for all three series considered. These results and those mentioned above all suggest that $\hat{\gamma}_{2,1}(k)$ converges to its asymptotic distribution more quickly than the BDS statistic does.

Results reported in Table 28 suggest that the Hinich linearity test converges to the normal distribution by sample size 512. For an independently and identically distributed normal sequence of sample size 256, the 80% Quantile Measure of $(2|\hat{\Gamma}(\omega_1,\omega_2)|)$ rejects slightly more often than it should. At a nominal size of .05, the estimated size is .06 with smoothing constant M = 12 and .075 with M = 17. Thus, for the independently and identically distributed normal case $\hat{\gamma}_{2,1}(k)$ converges to its asymptotic distribution more quickly than the Hinich linearity test. Ashley, Patterson and Hinich (1986) did not report simulation results on the estimated size of the Hinich linearity for any stochastic processes other than the independently and identically distributed normal.

G. A Portmanteau Version of the Test Statistics

The last topic I address in this chapter is a portmanteau version of time irreversibility test statistics. Recall the Monte Carlo results discussed above which suggested that the $\hat{\gamma}_{2,1}(k)$ values are uncorrelated across k. These results, along with the asymptotic normality of $\hat{\gamma}_{2,1}(k)$ for all k, motivate the following portmanteau test statistic which provides a joint test on a set of $\hat{\gamma}_{2,1}(k)$ values. I define:

$$P_{m,n} = \sum_{k=m}^{n} [\hat{\gamma}_{2,1}(k)]^2 \tag{4.14}$$

The results stated above imply that as a sum of the squares of uncorrelated normal random variables, $P_{m,n}$ is distributed $\chi^2(n\text{-}m)$.

Table 29 reports Monte Carlo results on the distribution of two versions of $P_{m,n}$, $P_{1,5}$ and $P_{1,10}$, for four independently and identically distributed cases and for Gaussian AR(1) residuals. In each simulation, 10000 observations on $P_{1,5}$ and $P_{1,10}$ were generated and Kolmogorov-Smirnov goodness of fit statistics were calculated to test the null hypotheses that $P_{1,5}$ and $P_{1,10}$ were distributed $\chi^2(5)$ and $\chi^2(10)$. At sample size 250, the chisquare hypotheses can be rejected at the 1% significance level for only two out of ten cases, $P_{1,5}$ for the standard exponential case and $P_{1,10}$ for the $\chi^2(1)$ case. By sample size 500, the chisquare hypotheses can not rejected for any case up to even the 20% significance level.

The portmanteau statistic $P_{m,n}$ provides a useful disgnostic to use along with the estimated symmetric-bicovariance function. As a first step it seems reasonable to test the hypothesis that a set of $\hat{\gamma}_{2,1}(k)$ at various values of k are jointly significanly different from zero with $P_{m,n}$. If $P_{m,n}$ rejects, then one proceeds to examine the pattern of rejection through the individual $\hat{\gamma}_{2,1}(k)$ values. The relation between $P_{m,n}$ and a set of $\hat{\gamma}_{2,1}(k)$ values is exactly analogous to the relation between the Q statistic and the estimated autocorrelation function.

5. Estimating Power

I ran many Monte Carlo simulations in order to study the power of the time irreversibility test statistic $\hat{\gamma}_{2,1}(k)$. The two classes of models chosen to study were a bilinear BL(0,0,1,1) model and a threshold autoregressive TAR(1) model. For each class of model, the power was studied as both sample size and model parameters varied. In addition, a set of simulations was run in order to compare the power of $\hat{\gamma}_{2,1}(k)$ and $P_{m,n}$ against the power of the BDS statistic and Hinich's bispectrum linearity test.

Before getting into details, the results can be summarized as follows. For all TAR(1) models studied, the estimated power is high only at lag $k = 1$. For most of the BL(0,0,1,1) models considered, past lag $k = 2$ the estimated power seems to decline exponentially as the lag k increases. $\hat{\gamma}_{2,1}(k)$ is shown to be as equally or more powerful as the BDS statistic for a particular TAR(1) model studied by Hsieh and LeBaron; the power of the BDS statistics varies as the scaling parameter varies. As noted above, however, for finite sample size the true α-levels for the BDS statistic are often far greater than their nominal levels. Finally, $\hat{\gamma}_{2,1}(k)$ and the portmanteau statistics have greater estimated power than Hinich's test against a TAR(1) alternative but lower estimated power than Hinich's test against a bilinear alternative.

A. Estimated Power of $\hat{\gamma}_{2,1}(k)$ Against Two Classes of Alternatives

Tables 30 through 32 present results for the following bilinear model:

$$X_t = \beta \cdot X_{t-1} \cdot \epsilon_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0,1) \qquad (5.1)$$

for $\beta = .9,.8,\ldots,.1$, at sample size 100, 250 and 500, respectively. As expected, the estimated probability of rejection increases as sample size increases. With the exception of lag $k = 1$, at each sample size the power is a declining function of the parameter β. In so far as the coefficient β can be interpreted as an index of time irreversibility, then the results show that the power of the test increases as the degree of irreversibility increases. An interesting pattern emerges at lag $k = 1$. As β goes from .9 to .6, the power decreases at all sample sizes. But the power at $\beta = .5,.4,.3,.2$, is greater than it is at $\beta = .6$. This is true at all sample sizes. At the first lag, the power appears to be greatest at $\beta = .4$ and $\beta = .3$. For sample size 250, the probability of rejection at lag $k = 1$ is greater than .9 at both $\beta = .4$ and $\beta = .3$.

Tables 33 through 35 present results for the following threshold autoregressive model:

$$X_t = \alpha \cdot X_{t-1} + \epsilon_t, \text{ if } X_{t-1} \geq 1$$
$$X_t = -.4 \cdot X_{t-1} + \epsilon_t, \text{ if } X_{t-1} < 1, \; \epsilon_t \sim N(0,1) \quad (5.2)$$

for $\alpha = -.9,-.8,\ldots,-.1$. Note that for $\alpha = .4$, (5.2) is a standard AR(1) model. The following feature of the TAR(1) model stands out in these tables. That is, significant rejections show up only at lag $k = 1$. For almost all sample sizes and almost all parameter values, the probability of rejection at lag $k \neq 1$ is not far from the nominal value of .05. In contrast, for the bilinear model (5.1) significant rejections also show up at lag $k = 2,3,4,5$. Interpreting the quantity $|\alpha + .4|$ as an index of time irreversibility for this model, then at lag $k = 1$, the power of the test unambiguously increases as the degree of time irreversibility increases at all sample sizes. Note that when the value of the index equals zero, i.e., $\alpha = -.4$ and the model collapses to a conventional AR(1) model, the probability of a rejection at lag $k = 1$ is roughly equal to the nominal size of .05.

B. Estimated Power of Portmanteau Statistics Against Two Alternatives

Letting $\beta = .9$ for the bilinear model and $\alpha = -.9$ for the threshold autoregressive model, I also ran Monte Carlo simulations to estimate the power of the portmanteau statistics $P_{1,5}$ and $P_{1,10}$. For the bilinear model, $P_{1,5}$ appears to have more power than $P_{1,10}$, reaching a probability of rejection equal to .88 at sample size 500. Comparing with results presented in Tables 30-33, it is seen that the portmanteau statistics have greater estimated power than $\hat{\gamma}_{2,1}(k)$ at any single lag k. Recall that in contrast to the TAR(1) models studied, significant rejections for the BL(0,0,1,1) models occur at lags other than just $k = 1$. This is the apparent cause of the increase in estimated power brought about by performing a joint test through the portmanteau statistics.

On the other hand, the estimated power of $P_{1,5}$ and $P_{1,10}$ is less than the estimated power of $\hat{\gamma}_{2,1}(k)$ at lag $k = 1$ for the TAR(1) model at all sample sizes. This is consistent with the results presented in Tables 33-35 in which significant rejections occur only at lag $k = 1$ for this model. For this model neither $P_{1,5}$ nor $P_{1,10}$ seems to have more power than the other, in contrast to the bilinear case. At sample size 500, they reach a power of .52 and .57, respectively.

C. Power Comparison with BDS Test

Hsieh and LeBaron (1988) studied the power of the BDS statistic for the following threshold autoregressive model:

$$\begin{aligned} X_t &= .5 \cdot X_{t-1} + \epsilon_t, \text{ if } X_{t-1} \geq 1 \\ X_t &= -.4 \cdot X_{t-1} + \epsilon_t, \text{ if } X_{t-1} < 1, \ \epsilon_t \sim N(0,1) \end{aligned} \tag{5.3}$$

The results at nominal size equal to .05 are found below in Table 37. At low values of the scaling parameter for embedding dimension 2, the power of the BDS statistic is greater than .9 by sample size 500. Recall, however, the results on the estimated sizes of the BDS statistic. More specifically, at sample size 500 for all values of r/σ the true α-values are greater than the nominal level of .05 in the independently and identically distributed N(0,1) case. For the two lowest values of the scaling parameter, $r/\sigma = 0.25$ and $r/\sigma = 0.50$, the estimated sizes of the statistic were approximately five and two times its nominal level. At sample size 1000, the power reaches 1.00 at these lower values of the scaling parameter for embedding dimension 2.

Table 38 reports results on the power of $\hat{\gamma}_{2,1}(k)$. Above I noted that for TAR(1) models, significant rejections appear to show up only at lag $k = 1$. Restricting attention to this lag, the estimated power of $\hat{\gamma}_{2,1}(k)$ compares very well to the estimated power of the BDS statistic. By sample size 500, the probability of rejection at lag $k = 1$ is equal to .97. The estimated power of the BDS statistics is slightly higher, at .98, only for $r/\sigma = 0.50$. However, recall that the true α-level for the BDS statistics, at sample size 500 and at $r/\sigma = 0.50$, is most likely significantly higher. If the nominal size for $\hat{\gamma}_{2,1}(k)$ at lag $k = 1$ were set higher, chances are good that the estimated power would also increase. At sample size 500, the estimated power of $\hat{\gamma}_{2,1}(k)$ at lag $k = 1$ is greater than the power of the BDS statistic for values of r/σ greater than 0.50. By sample size 1000, the estimated probability of rejection for $\hat{\gamma}_{2,1}(k)$ at lag $k = 1$ is 1.00. This matches the estimated power of the BDS statistic for low values of r/σ at this sample size and is greater than the estimated power for $r/\sigma = 1.50$ and $r/\sigma = 2.00$.

Thus, ignoring the high estimated sizes of the BDS statistic at low values of r/σ, $\hat{\gamma}_{2,1}(k)$ at lag $k = 1$ is found to have equal or greater estimated power than the BDS statistic. At low values of r/σ, it has equal power. At higher values of r/σ, its estimated power is greater. On the other hand, if the effectively low nominal levels for low values of r/σ are taken into consideration, $\hat{\gamma}_{2,1}(k)$ at lag $k = 1$ is found to be more powerful than the BDS statistic across all values of the scaling parameter.

D. Power Comparison with Hinich Test

Ashley, Patterson and Hinich (1986) studied the power of the following threshold autoregressive model:

$$\begin{aligned} X_t &= -.5 \cdot X_{t-1} + \epsilon_t, \text{ if } X_{t-1} \geq 1 \\ X_t &= .4 \cdot X_{t-1} + \epsilon_t, \text{ if } X_{t-1} < 1, \ \epsilon_t \sim N(0,1) \end{aligned} \tag{5.4}$$

They also studied the power of the following bilinear model:

$$X_t = .7 \cdot X_{t-1} \cdot \epsilon_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0,1) \qquad (5.5)$$

Their results for the Hinich linearity test appear below in Table 39. By sample size 500, the test rejects with probability .55 for the TAR(1) model and with probability .96 for the BL(0,0,1,1) model. By sample size 100, the probabilities reach .80 and 1.00 for the two models.

Table 40 reports results on the power of $\hat{\gamma}_{2,1}(k)$ for the TAR(1) model (5.4). Recall once again the significant rejections for this class of model show up just at lag $k = 1$. At this lag, the probability of rejection equals .997 and 1.00 at sample size 500 and 1000, respectively. Thus, the power of $\hat{\gamma}_{2,1}(1)$ is greater than the power of Hinich's linearity test for this TAR(1) model. Results for the portmanteau statistics are found in Table 41. The probability of rejection of $P_{1,5}$ is equal to .98 and 1.00 at sample size 500 and 1000, respectively. So $P_{1,5}$ is also more powerful than Hinich's linearity test for this model.

Results on the power of $\hat{\gamma}_{2,1}(k)$ for the BL(0,0,1,1) model (5.5) can be found in Tables 30 through 32. By sample size 500, the probability of rejection for $\hat{\gamma}_{2,1}(1)$ and $\hat{\gamma}_{2,1}(2)$ is equal to .46 and .74. Results for the portmanteau statistics can be found in Table 41. The probability of rejection of $P_{1,5}$ is equal to .78 and .94 at sample size 500 and 1000, respectively. Thus, Hinich's linearity test is more powerful than either $\hat{\gamma}_{2,1}(k)$ or $P_{1,5}$ for this model.

6. Testing Economic Time Series

I tested several time series for time irreversibility. The series examined were: (1) the Wolf sunspot series, 1700-1955; (2) the Canadian lynx data, 1821-1934; (3) the log first differences of quarterly nominal GNP, 1946:1-1988:4, as reported in the Citibase databank (172 observations); (4) the monthly aggregate unemployment rate, 1948:01-1989:01, as reported in the Citibase databank (492 observations); (5) the monthly manufacturing sector capacity utilization rate, 1948:01-1989:01, as reported in the Citibase databank (492 observations); (6) the log first differences of the monthly pigiron production series from 1877 to 1930 (637 observations); (7) the value weighted weekly stock return data from July 1962 to July 1984 as computed by Scheinkman and LeBaron (1987) on the Center for Research in Security Prices (CRSP) data (1227 observations); (8) weekly spot cotton prices from January 1972 to June 1986[16] as reported in the WEFA Group Financial Data Base (702 observations).

For each series, except for the stock returns data, I first fitted an ARMA model to the data. As is well known, the stock returns

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16 While I had data out to November 1988 for cotton prices, from the time series plot it was clear that a structural break occurred after June 1986. I consequently deleted all observations after June 1986.

series exhibits very little serial correlation. The sample autocorrelation functions for each set of residuals for the other datasets contained no evidence of serial correlation. I calculated the portmanteau statistics and estimated the symmetric-bicovariance function for these series on the residuals. For the stock returns series, I calculated portmanteau statistics and the symmetric-bicovariance function on the original data. The calculated portmanteau statistics are reported in Table 42. The estimated symmetric-bicovariance functions, together with the confidence intervals, appear in Figures 1-8. For all series the approximate 95% confidence intervals were formed by taking twice the theoretical value of $Var(\hat{\gamma}_{2,1}(k))^{1/2}$ for the independently and identically distributed case for the given sample size, using sample estimates of the third and fourth moments.

For each series I calculated the portmanteau statistics $P_{1,5}$, $P_{1,10}$, $P_{1,20}$ and $P_{1,30}$. The sunspot series rejects at the 5% significance level for each statistic while the lynx series does not reject for any statistic. Nominal GNP growth rate residuals reject for $P_{1,20}$ and $P_{1,30}$ but not for $P_{1,5}$ and $P_{1,10}$ at the 5% significance level. The aggregate unemployment rate rejects for each statistic while the capacity utilization rate rejects only for $P_{1,5}$ and $P_{1,10}$. The pigiron, cotton prices and stock returns series all reject for each of the statistics at the 5% significance level.

The estimated symmetric-bicovariance function for the ARMA(6,6) sunspot and ARMA(3,3) lynx residuals appear in Figure 1 and 2. Priestley (1984) reported that these ARMA representations give the lowest AIC values within a wide class of models considered. The time reversibility of sunspot series shows up in the first two lags. There is no evidence of time irreversibility in the lynx data in Figure 2. It may be the case that one must go to a higher version of $\hat{\gamma}_{i,j}(k)$, $i+j > 3$, in order to detect the time irreversibility in this series, or simply that the process of taking residuals from an ARMA(3,3) model lowered the power of out test. The reader should note that the confidence intervals are calculated as marginal intervals for each k; they are not joint, or simultaneous, confidence intervals for the whole sequence of $\hat{\gamma}_{2,1}(k)$, $k = 1,\ldots,n$.

The estimated symmetric-bicovariance for the residuals of three closely watched business cycle indicators, the quarterly growth rate in nominal GNP, the monthly aggregate unemployment rate, and the monthly manufacturing sector capacity utilization rate, appear in Figures 3 to 5. Several spikes appear outside the confidence intervals for all three series. There thus is evidence of Type 1 time irreversibility in all three series. To the extent that these variables are major business cycle indicators, these results suggest that the business cycle is indeed asymmetric.

A plot of the smoothed sample symmetric-bicovariance function for the MA(3) residuals of the log first differences of this series is presented in Figure 6. Most of the evidence for Type 1 time irreversibility appears to taper off after 12 months.

The estimated symmetric-bicovariance function for the two financial time series, the stock returns data and spot cotton prices, are plotted in Figures 7 and 8. There is strong evidence of Type 1 time irreversibility in both series. In the stock returns data, the evidence is concentrated in the first thirty weeks. Rejection of time reversibility is found at almost all lags up to fifty-five for the spot cotton prices.

7. Conclusions and Suggestions for Future Work

I have introduced a time domain test of time reversibility, the empirical symmetric-bicovariance function. The test can be used to provide a diagnostic check on the adequacy of the Frisch-type approach to modelling macroeconomic fluctuations. Several key business cycle indicators have been shown to be irreversible. This then implies that business cycle movements are asymmetric and, to the extent that these irreversibilities are important, both calls into question conventional time series techniques used in applied macroeconomics and suggests the need for macroeconomic theorists to develop state-dependent and regime switching models. Since it is known that all Gaussian ARMA processes are time reversible, it would be inappropriate to model these and other time irreversible series as Gaussian ARMA.

Under the null hypothesis the mean of the test statistic was shown to be equal to zero. For the independently and identically distributed case an exact small sample expression for the variance was derived. In this case the statistic was shown to be asymptotically distributed normal. An approximate expression for the variance in the ARMA case was obtained and this was shown to be large relative to the independently and identically distributed case. This then motivated the transformation to ARMA residuals in order to reduce the variance of the test statistic. Assuming the model has been correctly identified, the sampling distribution for the independently and identically distributed case can then be applied, as a useful approximation for large sample size; this is justified by the consistency of the estimates of the model's parameters.

The null hypothesis using the $\hat{\gamma}_{2,1}(k)$ statistic was restricted to joint probability distributions for which the first six moments are finite. Also, I restricted attention for practical reasons to $\hat{\gamma}_{2,1}(k)$ and did in general look at $\hat{\gamma}_{i,j}(k)$ for arbitrary (i,j). Thus, my results are conservative in that $\hat{\gamma}_{2,1}(k)$ may indicate acceptance, but $\hat{\gamma}_{i,j}(k)$ for $(i,j) \neq (2,1)$ may not. As noted above, this may indeed be the case for the Canadain lynx series. In future work I plan to test for time irreversibility with a more generalized version of $\hat{\gamma}_{i,j}(k)$.

I studied the small sample properties of the estimated symmetric-bicovariance function in Monte Carlo simulations. For several independently and identically distributed cases, convergence to the asymptotic distribution occurred by sample size 100. For AR(1) residuals, the estimated sizes were approximately double the nominal size at sample size 100. By sample size 250, the estimated sizes were just slightly greater than the nominal size. Convergence to the asymptotic normal distribution occurred by sample size 500. I showed that these rates of convergence compared very favorably to rates of convergence for both the BDS test and Hinich's linearity test. This was especially true for the BDS test.

Monte Carlo evidence was presented which suggested that the $\hat{\gamma}_{2,1}(k)$ values were uncorrelated across k. Given the asymptotic normality of the $\hat{\gamma}_{2,1}(k)$ values, this then motivated a portmanteau version of the test statistic. The distributions of the portmanteau statistics, chi-square with the appropriate degrees of freedom, were confirmed under the null hypothesis via Monte Carlo simulations. This then allows joint tests of significance to be carried out.

Through Monte Carlo simulations I estimated the power of both $\hat{\gamma}_{2,1}(k)$ and the portmanteau statistics against two classes of alternatives: BL(0,0,1,1) bilinear models and TAR(1) threshold autoregressive models. The pattern of time irreversibility, as revealed by the estimated symmetric-bicovariance function, was seen to vary across these classes of models. More specifically, the generic pattern for TAR(1) models is very high power at lag $k = 1$ but estimated probability of rejection at the nominal level at all other lags. For most of the BL(0,0,1,1) models considered, the generic pattern was high power at both lag $k = 1$ and $k = 2$ but exponentially declining power as the k lag increased past $k = 2$.

This then offers a simple diagnostic which can be used for identification of these time series models. By characterizing the time irreversibility through the estimated symmetric-bicovariance function, systematic patterns are revealed which assist the time series analyst in carrying out Box-Jenkins-like identification. This, I believe, represents an advance over the BDS and Hinich tests. As I stressed above, while those tests may be useful in rejecting the particular null hypothesis, results obtained do not serve as a direct guide to specification of an appropriate time series model.

For the TAR(1) models, the estimated power at lag $k = 1$ is greater than the estimated power of the portmanteau statistics. Given the low estimated power at all lags other than $k = 1$, this is not a surprising. For the BL(0,0,1,1) models, the portmanteau statistics generally are more powerful than $\hat{\gamma}_{2,1}(k)$ at any particular k lag by sample size 250.

The estimated power of the time irreversibility test statistics compared well when matched up against the BDS test and the Hinich test. For a TAR(1) alternative, $\hat{\gamma}_{2,1}(k)$ at lag $k = 1$ was shown to be generally more powerful than the BDS statistic at various values of the scaling parameter. For a different TAR(1) alternative, $\hat{\gamma}_{2,1}(k)$ at lag $k = 1$ was shown to have much greater estimated power than Hinich's test. However, Hinich's test was more powerful against a BL(0,0,1,1) alternative.

In the applications chapter I produced statistical evidence that the famous sunspot series is indeed time irreversible as are some macroeconomic and financial time series. The evidence of irreversibility suggests that policy decisions should take such factors into account and I have indicated the lag over which such irreversibilities are important. The fact that stock returns are time irreversible should prove of interest to researchers in finance who are interested in documenting time dependence in stock returns.

Two important lines of work lie ahead. First, I need to produce a taxonomy of mappings from various patterns in the estimated symmetric-bicovariance function to particular time series models which produce those patterns. As noted above, the power simulations already done suggest some representative patterns for TAR(1) and BL(0,0,1,1) models. One strategy to follow is to start with a simple Volterra expansion and calculated the symmetric bicovariance function. Second, work needs to be done to relate the notion of time irreversibility more directly to the current stock of macroeconomic theoretical models.

Appendix: Variance of Test Statistic in the MA(2) Case

Suppose $\{X_t\}$ is an MA(2) process, with moving average parameters b_1 and b_2. Then, an exact expression for the variance of $\hat{\gamma}_{2,1}(k)$, for $k \geq 5$, is:

$$2(T-k)/(T-k)^2 \; \{\mu_2^3(6b_1^2 + 6b_1^4 + 6b_2^2 + 18b_1^2b_2^2 + b_1^4b_2^2 + 6b_2^4 + 6b_1^2b_2^4) + \mu_2\mu_4(1 + b_1^2 + b_1^4 + b_1^6 + b_2^2 + b_1^4b_2^2 + b_2^4 + b_1^2b_2^4 + b_2^6)\} +$$

$$4(T-k-1)/(T-k)^2 \; \{\mu_2\mu_4(-b_1^3 + b_1^3b_2 - b_1^3b_2^2 + b_1^3b_2^3) + \mu_2^3(-b_1 - b_1^3 - b_1^5 + b_1b_2 + 5b_1^3b_2 + b_1^5b_2 - 2b_1b_2^2 - 5b_1^3b_2^2 + 2b_1b_2^3 + b_1^3b_2^3 - b_1b_2^4 + b_1b_2^5)\} +$$

$$4(T-k-2)/(T-k)^2 \; \{\mu_2\mu_4(-b_2^3) + \mu_2^3(-b_2 - 2b_1^2b_2 - b_1^4b_2 - b_2^3 - 2b_1^2b_2^3 - b_2^5)\} +$$

$$2(T-2k-2)/(T-k)^2 \; \{\mu_2^3(b_2 + 2b_1^2b_2 + b_1^4b_2 + 2b_2^3 + 2b_1^2b_2^3 + b_2^5)\} +$$

$$2(T-2k-1)/(T-k)^2 \; \{\mu_2^3(b_1 + 2b_1^3 + b_1^5 - b_1b_2 - 2b_1^3b_2 - b_1^5b_2 + 2b_1b_2^2 + 2b_1^3b_2^2 - 2b_1b_2^3 - 2b_1^3b_2^3 + b_1b_2^4 - b_1b_2^5)\} +$$

$$2(T-2k)/(T-k)^2 \; \{\mu_2^3(-1 - 3b_1^2 - 3b_1^4 - b_1^6 - 3b_2^2 - 6b_1^2b_2^2 - 3b_1^4b_2^2 - 3b_2^4 - 3b_1^2b_2^4 - b_2^6)\} +$$

$$2(T-2k+1)/(T-k)^2\ (\mu_2^3(b_1 + 2b_1^3 + b_1^5 - b_1b_2 - 2b_1^3b_2 - b_1^5b_2 + 2b_1b_2^2 + 2b_1^3b_2^2 - 2b_1b_2^3 - 2b_1^3b_2^3 + b_1b_2^4 - b_1b_2^5)) +$$

$$2(T-2k+2)/(T-k)^2\ (\mu_2^3(b_2 + 2b_1^2b_2 + b_1^4b_2 + 2b_2^3 + 2b_1^2b_2^3 + b_2^5))$$

Table 1

Some Summary Statistics on the Empirical Sampling Distribution of $\gamma_{2,1}(1)$: $\{X_t\}$ Independent and Identically Distributed N(0,1)

T[1]	Measured Kurtosis	Kolmogorov-Smirnov D Statistic for the Normal Distribution[2]	Theoretical Variance[3]	Estimated Variance
50	4.250	0.0415	0.287	0.281
100	4.159	0.0271	0.202	0.205
150	3.606	0.0205	0.164	0.168
200	3.335	0.0149	0.142	0.140
250	3.352	0.0176	0.127	0.127
300	3.370	0.0173	0.116	0.118
350	3.407	0.0195	0.107	0.110
400	3.193	0.0139	0.100	0.100
450	3.238	0.0137	0.094	0.097
500	3.025	0.0181	0.090	0.090

---

[1]T - sample size.

[2]For a sample size of 1000, the acceptance limits for the Kolmogorov-Smirnov Test of Goodness of Fit are:

Significance Level

.20	.15	.10	.05	.01
0.0338	0.0360	0.0386	0.0430	0.0515

Large values reject. See the asymptotic formula given in Lindgren (1976, p.580).

[3]As derived from equation (4.12).

These results are based on Monte Carlo simulations with 1000 iterations for each sample size. In each iteration, a series of length T of independently and identically variables, distributed normal with mean zero and unit variance, was generated and $\gamma_{2,1}(1)$ was calculated.

Table 2

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$:
(X_t) Independently and Identically Distributed N(0,1)

	T = 100 [1]	T = 250	T = 500	T = 1000	α Level [2]
k = 1 [3]	.048	.048	.049	.047	.05
k = 2	.051	.053	.051	.049	.05
k = 3	.049	.046	.052	.047	.05
k = 4	.052	.058	.056	.048	.05
k = 5	.058	.051	.043	.044	.05
k = 6	.048	.040	.059	.045	.05
k = 7	.049	.048	.051	.044	.05
k = 8	.047	.045	.049	.044	.05
k = 9	.048	.048	.046	.044	.05
k = 10	.044	.055	.050	.046	.05

[1]T = Sample size of (X_t).

[2]α = Probability of rejection under the null hypothesis of time reversible.

[3]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of independently and identically N(0,1) random variables was generated, $\gamma_{2,1}(k)$ was calculated and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 3

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ Independently and Identically Distributed $\chi^2(1)$

	T = 100 [1]	T = 250	T = 500	T = 1000	α Level [2]
k = 1 [3]	.052	.053	.046	.045	.05
k = 2	.048	.050	.050	.052	.05
k = 3	.045	.049	.047	.047	.05
k = 4	.047	.049	.047	.050	.05
k = 5	.046	.048	.046	.047	.05
k = 6	.049	.051	.049	.052	.05
k = 7	.052	.049	.046	.051	.05
k = 8	.049	.049	.047	.050	.05
k = 9	.052	.047	.048	.052	.05
k = 10	.047	.052	.046	.046	.05

[1] T = Sample size of $\{X_t\}$.

[2] α = Probability of rejection under the null hypothesis of time reversible.

[3] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of independently and identically $\chi^2(1)$ random variables was generated, $\hat{\gamma}_{2,1}(k)$ was calculated and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 4

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ Independently and Identically Distributed $\chi^2(5)$

	T = 100 [1]	T = 250	T = 500	T = 1000	α Level [2]
k = 1 [3]	.049	.045	.046	.046	.05
k = 2	.048	.046	.046	.044	.05
k = 3	.046	.045	.046	.049	.05
k = 4	.044	.045	.046	.045	.05
k = 5	.047	.044	.046	.044	.05
k = 6	.048	.047	.045	.044	.05
k = 7	.046	.044	.046	.046	.05
k = 8	.047	.047	.046	.045	.05
k = 9	.047	.046	.042	.043	.05
k = 10	.046	.047	.045	.043	.05

[1] T = Sample size of $\{X_t\}$.

[2] α = Probability of rejection under the null hypothesis of time reversible.

[3] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of independently and identically $\chi^2(5)$ random variables was generated, $\hat{\gamma}_{2,1}(k)$ was calculated and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 5

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ Independently and Identically Distributed Standard Exponential

	T = 100 [1]	T = 250	T = 500	T = 1000	α Level [2]
k = 1 [3]	.050	.042	.047	.044	.05
k = 2	.047	.047	.046	.046	.05
k = 3	.045	.045	.046	.044	.05
k = 4	.047	.046	.044	.046	.05
k = 5	.045	.046	.047	.043	.05
k = 6	.044	.044	.046	.042	.05
k = 7	.042	.044	.048	.049	.05
k = 8	.044	.042	.044	.043	.05
k = 9	.042	.043	.046	.042	.05
k = 10	.046	.047	.046	.049	.05

[1] T = Sample size of $\{X_t\}$.

[2] α = Probability of rejection under the null hypothesis of time reversible.

[3] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of independently and identically standard exponential random variables was generated, $\hat{\gamma}_{2,1}(k)$ was calculated and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 6

Probability that Number of Rejections is Greater Than or Equal to k, k = 1,2,...,10: $\{X_t\}$ Independently and Identically Distributed N(0,1)

	T = 100 [1]	T = 250	T = 500	T = 1000	Theoretical Probability [2]
k = 1 [3]	.407	.384	.374	.373	.401
k = 2	.094	.073	.074	.074	.086
k = 3	.014	.011	.008	.010	.012
k = 4	.001	.002	.001	.001	.001
k = 5	0.000	0.000	0.000	0.000	0.000

[1]T = Sample size of $\{X_t\}$.

[2]Probability of rejecting k or more times in a sequence of Bernoulli trials in which the probablity of success is .05.

[3]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of independently and identically N(0,1) random variables was generated, $\hat{\gamma}_{2,1}(k)$ was calculated and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 7

Probability that Number of Rejections is
Greater Than or Equal to k, k = 1,2,...,10:
$\{X_t\}$ Independently and Identically Distributed $\chi^2(1)$

	T = 100 [1]	T = 250	T = 500	T = 1000	Theoretical Probability [2]
k = 1 [3]	.377	.377	.372	.369	.401
k = 2	.080	.070	.072	.070	.086
k = 3	.011	.009	.009	.010	.012
k = 4	.001	.001	.001	.001	.001
k = 5	0.000	0.000	0.000	0.000	0.000

[1] T = Sample size of $\{X_t\}$.

[2] Probability of rejecting k or more times in a sequence of Bernoulli trials in which the probablity of success is .05.

[3] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of independently and identically $\chi^2(1)$ random variables was generated, $\gamma_{2,1}(5)$ was calculated and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 8

Probability that Number of Rejections is Greater Than or Equal to k, k = 1,2,...,10: $\{X_t\}$ Independently and Identically Distributed $\chi^2(5)$

	T = 100 [1]	T = 250	T = 500	T = 1000	Theoretical Probability [2]
k = 1 [3]	.385	.399	.389	.393	.401
k = 2	.089	.085	.074	.087	.086
k = 3	.012	.012	.009	.011	.012
k = 4	.001	.001	.001	.001	.001
k = 5	0.000	0.000	0.000	0.000	0.000

[1] T = Sample size of $\{X_t\}$.

[2] Probability of rejecting k or more times in a sequence of Bernoulli trials in which the probablity of success is .05.

[3] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of independently and identically $\chi^2(1)$ random variables was generated, $\hat{\gamma}_{2,1}(k)$ was calculated and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 9

Probability that Number of Rejections is
Greater Than or Equal to k, k = 1,2,...,10:
$\{X_t\}$ Independently and Identically Distributed Standard Exponential

	T = 100 [1]	T = 250	T = 500	T = 1000	Theoretical Probability [2]
k = 1 [3]	.364	.363	.373	.366	.401
k = 2	.075	.074	.075	.072	.086
k = 3	.010	.008	.011	.008	.012
k = 4	.001	.001	.001	.001	.001
k = 5	0.000	0.000	0.000	0.000	0.000

[1] T = Sample size of $\{X_t\}$.

[2] Probability of rejecting k or more times in a sequence of Bernoulli trials in which the probablity of success is .05.

[3] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of independently and identically standard exponential random variables was generated, $\gamma_{2,1}(k)$ was calculated and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 10

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.9)

	Estimated Sizes, α Level[1] = 0.05			**Prob (Rejecting k or More Times)**			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.102	.056	.050	.684	.462	.412	.401
k = 2	.104	.059	.047	.312	.119	.094	.086
k = 3	.112	.057	.052	.096	.019	.013	.012
k = 4	.111	.063	.049	.020	.002	.001	.001
k = 5	.113	.060	.054	.020	0.000	0.000	0.000
k = 6	.121	.059	.058	.004	0.000	0.000	0.000
k = 7	.112	.057	.056	.001	0.000	0.000	0.000
k = 8	.116	.061	.049	0.000	0.000	0.000	0.000
k = 9	.115	.062	.053	0.000	0.000	0.000	0.000
k = 10	.111	.063	.052	0.000	0.000	0.000	0.000

[1] α = Probability of rejection under the null hypothesis of time reversible.

[2] T = Sample size of $\{X_t\}$.

[3] Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.9 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 11

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.8)

	Estimated Sizes, α Level[1] = 0.05			**Prob (Rejecting k or More Times)**			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.102	.055	.054	.695	.470	.418	.401
k = 2	.112	.059	.051	.311	.125	.096	.086
k = 3	.114	.060	.054	.099	.021	.014	.012
k = 4	.109	.063	.053	.022	.002	.002	.001
k = 5	.111	.064	.051	.004	0.000	0.000	0.000
k = 6	.111	.064	.059	0.000	0.000	0.000	0.000
k = 7	.114	.062	.051	0.000	0.000	0.000	0.000
k = 8	.116	.064	.053	0.000	0.000	0.000	0.000
k = 9	.115	.063	.052	0.000	0.000	0.000	0.000
k = 10	.121	.064	.053	0.000	0.000	0.000	0.000

[1]α = Probability of rejection under the null hypothesis of time reversible.

[2]T = Sample size of $\{X_t\}$.

[3]Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.8 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 12

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.7)

	Estimated Sizes, α Level[1] = 0.05			**Prob (Rejecting k or More Times)**			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.105	.059	.052	.697	.469	.417	.401
k = 2	.108	.057	.055	.319	.123	.092	.086
k = 3	.117	.064	.052	.100	.018	.012	.012
k = 4	.114	.063	.049	.022	.002	.002	.001
k = 5	.115	.063	.047	.005	0.000	0.000	0.000
k = 6	.113	.062	.052	0.000	0.000	0.000	0.000
k = 7	.116	.063	.053	0.000	0.000	0.000	0.000
k = 8	.118	.061	.055	0.000	0.000	0.000	0.000
k = 9	.119	.062	.052	0.000	0.000	0.000	0.000
k = 10	.117	.059	.050	0.000	0.000	0.000	0.000

[1]α = Probability of rejection under the null hypothesis of time reversible.

[2]T = Sample size of $\{X_t\}$.

[3]Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.7 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 13

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.6)

	Estimated Sizes, α Level[1] = 0.05			Prob (Rejecting k or More Times)			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.103	.058	.050	.688	.472	.416	.401
k = 2	.107	.062	.051	.310	.127	.097	.086
k = 3	.116	.064	.053	.097	.020	.014	.012
k = 4	.113	.063	.055	.021	.002	.001	.001
k = 5	.115	.062	.056	.003	0.000	0.000	0.000
k = 6	.111	.064	.053	0.000	0.000	0.000	0.000
k = 7	.111	.064	.054	0.000	0.000	0.000	0.000
k = 8	.114	.059	.052	0.000	0.000	0.000	0.000
k = 9	.112	.062	.047	0.000	0.000	0.000	0.000
k = 10	.117	.064	.051	0.000	0.000	0.000	0.000

[1] α = Probability of rejection under the null hypothesis of time reversible.

[2] T = Sample size of $\{X_t\}$.

[3] Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.6 was generated, an AR(1) model was fitted to the series, $\hat{\gamma}_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 14

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.5)

	Estimated Sizes, α Level[1] = 0.05			Prob (Rejecting k or More Times)			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.107	.060	.052	.693	.467	.407	.401
k = 2	.117	.065	.051	.319	.126	.089	.086
k = 3	.114	.059	.053	.102	.023	.013	.012
k = 4	.119	.060	.050	.020	.003	.002	.001
k = 5	.115	.064	.051	.003	0.000	0.000	0.000
k = 6	.117	.065	.051	0.000	0.000	0.000	0.000
k = 7	.110	.058	.051	0.000	0.000	0.000	0.000
k = 8	.112	.062	.050	0.000	0.000	0.000	0.000
k = 9	.114	.066	.050	0.000	0.000	0.000	0.000
k = 10	.114	.061	.053	0.000	0.000	0.000	0.000

[1]α = Probability of rejection under the null hypothesis of time reversible.

[2]T = Sample size of $\{X_t\}$.

[3]Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.5 was generated, an AR(1) model was fitted to the series, $\hat{\gamma}_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 15

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.4)

	Estimated Sizes, α Level[1] = 0.05			**Prob (Rejecting k or More Times)**			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.110	.060	.049	.690	.476	.414	.401
k = 2	.118	.062	.052	.314	.131	.092	.086
k = 3	.114	.065	.055	.102	.022	.012	.012
k = 4	.111	.062	.052	.023	.002	.002	.001
k = 5	.113	.067	.053	.004	0.000	0.000	0.000
k = 6	.111	.064	.049	0.000	0.000	0.000	0.000
k = 7	.11	.061	.055	0.000	0.000	0.000	0.000
k = 8	.113	.066	.050	0.000	0.000	0.000	0.000
k = 9	.113	.063	.050	0.000	0.000	0.000	0.000
k = 10	.112	.062	.055	0.000	0.000	0.000	0.000

[1] α = Probability of rejection under the null hypothesis of time reversible.

[2] T = Sample size of $\{X_t\}$.

[3] Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.4 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 16

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.3)

	Estimated Sizes, α Level[1] = 0.05			**Prob (Rejecting k or More Times)**			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.107	.058	.053	.699	.471	.409	.401
k = 2	.114	.060	.055	.321	.128	.097	.086
k = 3	.121	.062	.055	.097	.021	.014	.012
k = 4	.115	.061	.050	.022	.002	.001	.001
k = 5	.119	.066	.052	.004	0.000	0.000	0.000
k = 6	.111	.059	.055	0.000	0.000	0.000	0.000
k = 7	.111	.062	.051	0.000	0.000	0.000	0.000
k = 8	.116	.066	.051	0.000	0.000	0.000	0.000
k = 9	.113	.063	.049	0.000	0.000	0.000	0.000
k = 10	.116	.066	.053	0.000	0.000	0.000	0.000

[1] α = Probability of rejection under the null hypothesis of time reversible.

[2] T = Sample size of $\{X_t\}$.

[3] Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4] k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.3 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 17

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that
Number of Rejections is Greater Than or Equal to k:
$\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.2)

	Estimated Sizes, α Level[1] = 0.05			**Prob (Rejecting k or More Times)**			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.112	.064	.050	.692	.471	.418	.401
k = 2	.110	.062	.056	.316	.123	.094	.086
k = 3	.111	.062	.055	.094	.020	.014	.012
k = 4	.112	.060	.051	.020	.003	.001	.001
k = 5	.119	.061	.053	.004	0.001	0.000	0.000
k = 6	.115	.061	.054	0.000	0.000	0.000	0.000
k = 7	.114	.063	.052	0.000	0.000	0.000	0.000
k = 8	.113	.061	.051	0.000	0.000	0.000	0.000
k = 9	.111	.065	.051	0.000	0.000	0.000	0.000
k = 10	.109	.059	.055	0.000	0.000	0.000	0.000

[1]α = Probability of rejection under the null hypothesis of time reversible.

[2]T = Sample size of $\{X_t\}$.

[3]Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.2 was generated, an AR(1) model was fitted to the series, $\hat{\gamma}_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 18

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$ and Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.1)

	Estimated Sizes, α Level[1] = 0.05			Prob (Rejecting k or More Times)			
	T=100[2]	T=250	T=500	T=100	T=250	T=500	Theor.[3] Prob.
k = 1[4]	.110	.062	.049	.696	.469	.418	.401
k = 2	.117	.068	.051	.326	.132	.094	.086
k = 3	.116	.061	.054	.100	.023	.011	.012
k = 4	.118	.064	.051	.023	0.000	.001	.001
k = 5	.118	.060	.055	.003	0.001	0.000	0.000
k = 6	.111	.059	.053	0.000	0.000	0.000	0.000
k = 7	.116	.061	.052	0.000	0.000	0.000	0.000
k = 8	.116	.063	.053	0.000	0.000	0.000	0.000
k = 9	.109	.063	.055	0.000	0.000	0.000	0.000
k = 10	.116	.065	.051	0.000	0.000	0.000	0.000

[1]α = Probability of rejection under the null hypothesis of time reversible.

[2]T = Sample size of $\{X_t\}$.

[3]Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[4]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with N(0,1) innovations and AR(1) coefficient equal to 0.1 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 19

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ Residuals from AR(1) with $\chi^2(1)$ Innovations
(AR(1) Coefficient = 0.9)

	T=100[1]	T=250	T=500	T=1000	T=5000	α Level [2]
k = 1[3]	.166	.083	.080	.080	.054	.05
k = 2	.141	.088	.068	.068	.051	.05
k = 3	.130	.083	.055	.061	.050	.05
k = 4	.121	.081	.067	.061	.062	.05
k = 5	.131	.065	.054	.065	.047	.05
k = 6	.113	.061	.071	.046	.051	.05
k = 7	.127	.069	.072	.056	.058	.05
k = 8	.106	.077	.049	.048	.043	.05
k = 9	.101	.073	.054	.049	.044	.05
k = 10	.129	.062	.069	.051	.053	.05

[1]T = Sample size of $\{X_t\}$.

[2]α = Probability of rejection under the null hypothesis of time reversible.

[3]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with $\chi^2(1)$ innovations and AR(1) coefficient equal to 0.9 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 20

Estimated Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Residuals from AR(1) with $\chi^2(1)$ Innovations (AR(1) Coefficient = 0.9)

	T=100[1]	T=250	T=500	T=1000	T=5000	Theoretical Probability[2]
k = 1[3]	.691	.523	.462	.432	.413	.401
k = 2	.350	.173	.138	.122	.090	.086
k = 3	.142	.036	.032	.027	.018	.012
k = 4	.056	.009	.006	.004	.002	.001
k = 5	.008	.001	.001	0.000	0.000	0.000
k = 6	0.000	0.000	0.000	0.000	0.000	0.000

[1]T = Sample size of $\{X_t\}$.

[2]Probability of rejecting k or more times in a sequence of Bernoulli trials in which the probablity of success is .05.

[3]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with $\chi^2(1)$ innovations and AR(1) coefficient equal to 0.9 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 21

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ Residuals from AR(1) with $\chi^2(5)$ Innovations
(AR(1) Coefficient = 0.9)

	T=100[1]	T=250	T=500	T=1000	T=5000	α Level [2]
k = 1[3]	.129	.055	.062	.062	.043	.05
k = 2	.108	.061	.072	.050	.056	.05
k = 3	.115	.064	.050	.051	.044	.05
k = 4	.122	.070	.062	.048	.050	.05
k = 5	.109	.066	.063	.065	.043	.05
k = 6	.109	.081	.042	.052	.044	.05
k = 7	.129	.062	.045	.061	.035	.05
k = 8	.133	.073	.042	.053	.055	.05
k = 9	.127	.043	.064	.058	.041	.05
k = 10	.121	.082	.054	.050	.050	.05

[1]T = Sample size of $\{X_t\}$.

[2]α = Probability of rejection under the null hypothesis of time reversible.

[3]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with $\chi^2(5)$ innovations and AR(1) coefficient equal to 0.9 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 22

Estimated Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Residuals from AR(1) with $\chi^2(5)$ Innovations (AR(1) Coefficient = 0.9)

	T=100[1]	T=250	T=500	T=1000	T=5000	Theoretical Probability[2]
k = 1[3]	.695	.507	.435	.434	.379	.401
k = 2	.349	.130	.099	.099	.064	.086
k = 3	.113	.017	.018	.015	.007	.012
k = 4	.032	.003	.003	.002	.001	.001
k = 5	.010	0.000	.001	0.000	0.000	0.000
k = 6	.003	0.000	0.000	0.000	0.000	0.000

[1]T = Sample size of $\{X_t\}$.

[2]Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[3]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with $\chi^2(5)$ innovations and AR(1) coefficient equal to 0.9 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 23

Estimated Sizes of $\hat{\gamma}_{2,1}(k)$: $\{X_t\}$ Residuals from AR(1) with Standard Exponential Innovations (AR(1) Coefficient = 0.9)

	T=100[1]	T=250	T=500	T=1000	T=5000	α Level [2]
k = 1[3]	.130	.076	.066	.061	.051	.05
k = 2	.129	.075	.068	.058	.054	.05
k = 3	.117	.072	.062	.056	.054	.05
k = 4	.121	.072	.059	.057	.055	.05
k = 5	.120	.067	.060	.059	.052	.05
k = 6	.116	.069	.060	.055	.045	.05
k = 7	.111	.067	.059	.051	.048	.05
k = 8	.119	.069	.055	.052	.053	.05
k = 9	.121	.062	.057	.054	.049	.05
k = 10	.112	.066	.056	.052	.047	.05

[1]T = Sample size of $\{X_t\}$.

[2]α = Probability of rejection under the null hypothesis of time reversible.

[3]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with standard exponential innovations and AR(1) coefficient equal to 0.9 was generated, an AR(1) model was fitted to the series, $\hat{\gamma}_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 24

Estimated Probability that Number of Rejections is Greater Than or Equal to k: $\{X_t\}$ Residuals from AR(1) with Standard Exponential Innovations (AR(1) Coefficient = 0.9)

	T=100[1]	T=250	T=500	T=1000	T=5000	Theoretical Probability[2]
k = 1[3]	.691	.503	.458	.437	.410	.401
k = 2	.338	.150	.120	.105	.093	.086
k = 3	.123	.034	.021	.017	.013	.012
k = 4	.035	.006	.003	.002	.001	.001
k = 5	.008	.001	0.000	0.000	0.000	0.000
k = 6	.001	0.000	0.000	0.000	0.000	0.000

[1]T = Sample size of $\{X_t\}$.

[2]Probability of rejecting k or more times in a sequence of 10 Bernoulli trials in which the probablity of success is .05.

[3]k = lag at which $\hat{\gamma}_{2,1}(k)$ was evaluated

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, an AR(1) series of length T with $\chi^2(1)$ innovations and AR(1) coefficient equal to 0.9 was generated, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 25

Estimated Sizes of BDS Statistic:
$\{X_t\}$ Independently and Identically Distributed N(0,1)

	r/σ [1]					
	0.25	0.50	1.00	1.50	2.00	α Level[2]
D=2,T=100[3]	.574	.306	.145	.128	.163	.05
D=2,T=500	.256	.102	.068	.066	.074	.05
D=2,T=1000	.153	.070	.054	.049	.052	.05
D=5,T=500	.332	.070	.053	.058	.061	.05
D=5,T=1000	.206	.067	.054	.061	.061	.05

[1] r = Scaling Parameter
σ = Standard Deviation of the Series

[2] α = Probability of rejection under the null hypothesis of i.i.d.

[3] D = Embedding dimension and T = Sample size

The size of the BDS statistics based on Monte Carlo results reported in Hsieh and LeBaron (1988), Table 1.

Table 26

Estimated Sizes of BDS Statistic:
$\{X_t\}$ Independently and Identically Distributed $\chi^2(4)$

	r/σ [1]					
	0.25	0.50	1.00	1.50	2.00	α Level[2]
D=2, T=100[3]	.426	.196	.113	.109	.115	.05
D=2, T=500	.151	.069	.058	.059	.059	.05
D=2, T=1000	.101	.069	.069	.073	.062	.05
D=5, T=500	.522	.107	.057	.056	.058	.05
D=5, T=1000	.399	.088	.058	.056	.053	.05

[1] r = Scaling Parameter
σ = Standard Deviation of the Series

[2] α = Probability of rejection under the null hypothesis of i.i.d.

[3] D = Embedding dimension and T = Sample size

The size of the BDS statistics based on Monte Carlo results reported in Hsieh and LeBaron (1988), Table 4.

Table 27

Estimated Sizes of BDS Statistic:
$\{X_t\}$ Gaussian AR(1) Residuals (AR(1) Coefficient = 0.5)

r/σ [1]

	0.50	1.00	1.50	2.00	α Level[2]
D=2, T=100[3]	.311	.144	.124	.167	.05
D=2, T=500	.098	.069	.069	.077	.05
D=2, T=1000	.078	.062	.059	.064	.05
D=5, T=500	.249	.080	.077	.076	.05
D=5, T=1000	.127	.059	.057	.062	.05

[1] r = Scaling Parameter
σ = Standard Deviation of the Series

[2] α = Probability of rejection under the null hypothesis of i.i.d.

[3] D = Embedding dimension and T = Sample size

The size of the BDS statistics based on Monte Carlo results reported in Hsieh and LeBaron (1988), Table 13.

Table 28

Estimated Sizes of Hinich Linearity Test:
$\{X_t\}$ Independently and Identically Distributed N(0,1)

		80% Quantile[1]	α Level[2]
T[4] = 256	M[3] = 12	.060	.05
	M = 17	.075	.05
T = 512	M = 16	.052	.05
	M = 23	.050	.05
T = 1024	M = 23	.046	.05
	M = 33	.057	.05

[1]Size of 80% Quantile Measure of $\{2|\hat{\Gamma}(\omega_1,\omega_2)|\}$.

[2]α = Probability of rejection under the null hypothesis of linearity.

[3]M = Smoothing constant.

[4]T = Sample size.

The size of the 80% quantile measures based on Monte Carlo results reported in Ashley, Patterson and Hinich (1986), Table 2.

Table 29

Empirical Distributions of Portmanteau Statistics:
Kolmogorov-Smirnov Goodness of Fit D Statistics for $\chi^2(5)$ and $\chi^2(10)$

Series	T=100[1] $P_{1,5}$	T=100[1] $P_{1,10}$	T=250 $P_{1,5}$	T=250 $P_{1,10}$	T=500 $P_{1,5}$	T=500 $P_{1,10}$
I.I.D. N(0,1)	.0242[2]	.0302	.0090	.0131	.0073	.0099
I.I.D. $\chi^2(1)$	.0231	.0250	.0143	.0173	.0088	.0079
I.I.D. $\chi^2(5)$	.0230	.0268	.0142	.0135	.0069	.0107
I.I.D. Standard Exponential	.0227	.0223	.0237	.0154	.0080	.0088
Gaussian AR(1) Residuals[3]	.0112	.0203	.0074	.0134	.0076	.0094

[1]T = Sample size.

[2]For a sample size of 10000, the acceptance limits for the Kolmogorov-Smirnov Test of Goodness of Fit are:

Significance Level				
.20	.15	.10	.05	.01
0.0107	0.0114	0.0122	0.0136	0.0163

[3]AR(1) coefficient = .9.

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of the particular stochastic process was generated and the portmanteau statistics $P_{1,5}$ and $P_{1,10}$ were calculated. After the 10000 iterations, the Kolmogorov-Smirnov one sample goodness of fit D statistics were calculated to test the null hypothesis that the distrbutions of $P_{1,5}$ and $P_{1,10}$ were, respectively, $\chi^2(5)$ and $\chi^2(10)$.

Table 30

Estimated Power of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ MA(1) Residuals of a BL(0,0,1,1) Model at Sample Size 100

β	Pr(Reject) for k = 1	Pr(Reject) for k = 2	Pr(Reject) for k = 3	Pr(Reject) for k = 4	Pr(Reject) for k = 5
0.9	.428	.385	.152	.083	.053
0.8	.369	.351	.109	.061	.048
0.7	.359	.305	.080	.048	.034
0.6	.412	.231	.055	.042	.037
0.5	.539	.153	.048	.042	.039
0.4	.651	.094	.041	.043	.045
0.3	.611	.063	.047	.048	.046
0.2	.400	.054	.052	.047	.050
0.1	.150	.053	.053	.049	.050

$X_t = \beta \cdot X_{t-1} \cdot \epsilon_{t-1} + \epsilon_t$, $\epsilon_t \sim N(0,1)$

Probabilities based on Monte Carlo simulations with 10000 iterations in which a BL(0,0,1,1) series was generated with 100 observations, an MA(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 31

Estimated Power of $\hat{\gamma}_{2,1}(k)$:
(X_t) MA(1) Residuals of a BL(0,0,1,1) Model at Sample Size 250

β	Pr(Reject) for k = 1	Pr(Reject) for k = 2	Pr(Reject) for k = 3	Pr(Reject) for k = 4	Pr(Reject) for k = 5
0.9	.559	.587	.278	.156	.098
0.8	.475	.586	.208	.105	.065
0.7	.404	.539	.139	.067	.045
0.6	.494	.427	.082	.047	.038
0.5	.736	.280	.055	.043	.041
0.4	.901	.143	.044	.045	.041
0.3	.937	.071	.045	.048	.046
0.2	.801	.048	.045	.044	.045
0.1	.318	.049	.046	.048	.045

$X_t = \beta \cdot X_{t-1} \cdot \epsilon_{t-1} + \epsilon_t,\ \epsilon_t \sim N(0,1)$

Probabilities based on Monte Carlo simulations with 10000 iterations in which a BL(0,0,1,1) series was generated with 250 observations, an MA(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 32

Estimated Power of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ MA(1) Residuals of a BL(0,0,1,1) Model at Sample Size 500

β	Pr(Reject) for k = 1	Pr(Reject) for k = 2	Pr(Reject) for k = 3	Pr(Reject) for k = 4	Pr(Reject) for k = 5
0.9	.671	.736	.424	.240	.140
0.8	.575	.763	.328	.159	.086
0.7	.455	.744	.217	.098	.055
0.6	.553	.637	.116	.056	.040
0.5	.843	.449	.063	.043	.041
0.4	.981	.214	.045	.042	.043
0.3	.996	.088	.047	.046	.046
0.2	.981	.052	.046	.043	.048
0.1	.569	.043	.045	.045	.047

$X_t = \beta \cdot X_{t-1} \cdot \epsilon_{t-1} + \epsilon_t$, $\epsilon_t \sim N(0,1)$

Probabilities based on Monte Carlo simulations with 10000 iterations in which a BL(0,0,1,1) series was generated with 500 observations, an MA(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 33

Estimated Power of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ AR(1) Residuals of TAR(1) Model at Sample Size 100

α	Pr(Reject) for k = 1	Pr(Reject) for k = 2	Pr(Reject) for k = 3	Pr(Reject) for k = 4	Pr(Reject) for k = 5
-0.9	.223	.057	.059	.047	.049
-0.8	.160	.056	.053	.055	.051
-0.7	.114	.053	.052	.055	.056
-0.6	.084	.053	.049	.051	.049
-0.5	.061	.051	.053	.052	.048
-0.4	.051	.054	.052	.047	.047
-0.3	.060	.055	.049	.051	.049
-0.2	.075	.056	.049	.052	.052
-0.1	.095	.055	.049	.047	.047

$X_t = \alpha \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = -.4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

Probabilities based on Monte Carlo simulations with 10000 iterations in which a TAR(1) series was generated with 500 observations, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 34

Estimated Power of $\hat{\gamma}_{2,1}(k)$:
(X_t) AR(1) Residuals of TAR(1) Model at Sample Size 250

α	Pr(Reject) for k = 1	Pr(Reject) for k = 2	Pr(Reject) for k = 3	Pr(Reject) for k = 4	Pr(Reject) for k = 5
-0.9	.436	.061	.065	.047	.051
-0.8	.311	.057	.050	.050	.050
-0.7	.189	.051	.048	.046	.050
-0.6	.117	.048	.049	.051	.048
-0.5	.067	.046	.045	.048	.049
-0.4	.053	.048	.047	.048	.048
-0.3	.060	.051	.047	.048	.045
-0.2	.104	.054	.050	.045	.044
-0.1	.169	.054	.051	.046	.047

$X_t = \alpha \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = -.4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

Probabilities based on Monte Carlo simulations with 10000 iterations in which a TAR(1) series was generated with 250 observations, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 35

Estimated Power of $\hat{\gamma}_{2,1}(k)$:
$\{X_t\}$ AR(1) Residuals of TAR(1) Model at Sample Size 500

α	Pr(Reject) for k = 1	Pr(Reject) for k = 2	Pr(Reject) for k = 3	Pr(Reject) for k = 4	Pr(Reject) for k = 5
-0.9	.706	.073	.069	.049	.048
-0.8	.500	.064	.057	.049	.048
-0.7	.328	.053	.048	.046	.045
-0.6	.163	.050	.047	.048	.042
-0.5	.078	.047	.049	.044	.046
-0.4	.047	.048	.047	.047	.046
-0.3	.072	.045	.044	.047	.044
-0.2	.160	.047	.049	.044	.047
-0.1	.303	.049	.047	.051	.047

$X_t = \alpha \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = -.4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

Probabilities based on Monte Carlo simulations with 10000 iterations in which a TAR(1) series was generated with 500 observations, an AR(1) model was fitted to the series, $\gamma_{2,1}(k)$ was calculated on the residuals and a rejection was counted if the absolute value of $\gamma_{2,1}(k)$ was greater than twice the standard deviation of $\gamma_{2,1}(k)$ as given in Equation (4.12).

Table 36

Estimated Power of Portmanteau Statistics:
Threshold Autoregressive and Bilinear Models

		Threshold[1] Autoregressive Model	Bilinear Model[2]
T = 100	$P_{1,5}$	.191	.503
	$P_{1,10}$	.177	.407
T = 250	$P_{1,5}$	.340	.746
	$P_{1,10}$	.335	.654
T = 500	$P_{1,5}$	.518	.879
	$P_{1,10}$	.567	.821

[1] $X_t = -.9 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = -.4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

[2] $X_t = -.9 \cdot X_{t-1} \cdot \epsilon_{t-1} + \epsilon_t$, $\epsilon_t \sim N(0,1)$

[3] T = Sample size.

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of the particular stochastic process was generated, the portmanteau statistics $P_{1,5}$ and $P_{1,10}$ were calculated and a rejection was recorded if the observed values of $P_{1,5}$ and $P_{1,10}$ were significant at the 5% level.

Table 37

Estimated Power of BDS Statistic:
$\{X_t\}$ Threshold Autoregressive Model

	r/σ [1]					
	0.25	0.50	1.00	1.50	2.00	α Level[2]
D=2,T=100[3]	.674	.773	.516	.312	.248	.05
D=2,T=500	.969	.976	.932	.756	.471	.05
D=2,T=1000	1.00	1.00	1.00	.949	.721	.05
D=5,T=500	.816	.841	.801	.594	.366	.05
D=5,T=1000	.860	.986	.972	.857	.601	.05

[1] r = Scaling Parameter
σ = Standard Deviation of the Series

[2] α = Probability of rejection under the null hypothesis of i.i.d.

[3] D = Embedding dimension and T = Sample size

$X_t = .5 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = -.4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

The power of the BDS statistics based on Monte Carlo results reported in Hsieh and LeBaron (1988), Table 10.

Table 38

Estimated Power of $\hat{\gamma}_{2,1}(k)$ at Lag k:
(X_t) Threshold Autoregressive Model Studied by Hsieh and LeBaron

Lag k	T = 100 [1]	T = 250	T = 500	T = 1000
1	.349	.799	.967	1.00
2	.059	.069	.099	.164
3	.052	.037	.041	.060
4	.053	.038	.046	.038
5	.059	.039	.029	.045

[1]T = Sample size.

$X_t = .5 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = -.4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

Threshold autoregressive model is same as the one for results reported above in Table 37. Probabilities based on Monte Carlo simulations with 10000 iterations in which a TAR(1) series was generated for the given sample size, $\hat{\gamma}_{2,1}(k)$ was calculated on the series and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 39

Estimated Power of Hinich Linearity Test:
$\{X_t\}$ Bilinear and Threshold Autoregressive

	Bilinear[1] 80% Quantile	Threshold Autoregressive[2] 80% Quantile
T[1] = 256	.78	.33
T = 512	.96	.55
T = 1024	1.00	.80

---

[1] $X_t = .7 \cdot X_{t-1} \cdot \epsilon_{t-1} + \epsilon_t, \; \epsilon_t \sim N(0,1)$

[2] $X_t = -.9 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = -.4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

[3] T = Sample size.

These Monte Carlo results reported in Ashley, Pattterson and Hinich (1986), Table 3.

Table 40

Estimated Power of $\hat{\gamma}_{2,1}(k)$ at Lag k:
$\{X_t\}$ Threshold Autoregressive Model Studied
by Ashley, Patterson and Hinich

Lag k	T[1] = 100	T = 250	T = 500	T = 1000
1	.585	.920	.997	1.00
2	.043	.051	.072	.071
3	.048	.054	.037	.047
4	.052	.056	.041	.031
5	.050	.057	.054	.041

[1]T = Sample size.

$X_t = -.5 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = .4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

Threshold autoregressive model is same as the one for results reported above in Table 39. Probabilities based on Monte Carlo simulations with 10000 iterations in which a TAR(1) series was generated for the given sample size, $\hat{\gamma}_{2,1}(k)$ was calculated on the series and a rejection was counted if the absolute value of $\hat{\gamma}_{2,1}(k)$ was greater than twice the standard deviation of $\hat{\gamma}_{2,1}(k)$ as given in Equation (4.12).

Table 41

Estimated Power of Portmanteau Statistics:
Threshold Autoregressive and Bilinear Models
Studied by Ashley, Patterson and Hinich

		Threshold[1] Autoregressive Model	Bilinear Model[2]
T = 100	$P_{1,5}$	.384	.377
	$P_{1,10}$	.297	.276
T = 250	$P_{1,5}$	.736	.619
	$P_{1,10}$	.585	.503
T = 500	$P_{1,5}$	.979	.797
	$P_{1,10}$	.931	.700
T = 1000	$P_{1,5}$	1.000	.936
	$P_{1,10}$	.999	.886

[1] $X_t = -.5 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} \geq 1$

$X_t = .4 \cdot X_{t-1} + \epsilon_t$, if $X_{t-1} < 1$, $\epsilon_t \sim N(0,1)$

[2] $X_t = .7 \cdot X_{t-1} \cdot \epsilon_{t-1} + \epsilon_t$, $\epsilon_t \sim N(0,1)$

[3]T = Sample size.

These results are based on Monte Carlo simulations with 10000 iterations for each sample size. In each iteration, a series of length T of the particular stochastic process was generated, the portmanteau statistics $P_{1,5}$ and $P_{1,10}$ were calculated and a rejection was recorded if the observed values of $P_{1,5}$ and $P_{1,10}$ were significant at the 5% level.

Table 42

Portmanteau Statistics for Economic Time Series

Series	$P_{1,5}$[1]	$P_{1,10}$	$P_{1,20}$	$P_{1,30}$
Sunspot Data	26.03	37.69	59.09	61.72
Lynx Data	4.87	8.14		
Nominal GNP	4.83	10.56	33.67	58.80
Unemployment Rate	20.23	52.38	77.98	83.96
Capacity Util. Rate	21.85	23.40	24.82	37.32
Pigiron	81.86	98.75	116.32	121.44
Cotton Prices	22.80	58.91	99.11	192.97
Stock Returns	70.90	109.10	168.85	216.04

[1]The 5% significance level for the chisquare distribution are:

$\chi^2(5) = 11.1$, $\chi^2(10) = 18.3$, $\chi^2(20) = 31.4$ and $\chi^2(30) = 43.8$.

For each series, except the Stock Returns series, an ARMA model was first fitted to the data. Then the portmanteau statistics $P_{1,5}$, $P_{1,10}$, $P_{1,20}$ and $P_{1,30}$ were calculated on the ARMA residuals.

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Figure 1

Est. Symmetric-Bicovariance Funct. for

ARMA(6,6) Residuals of Sunspot Data

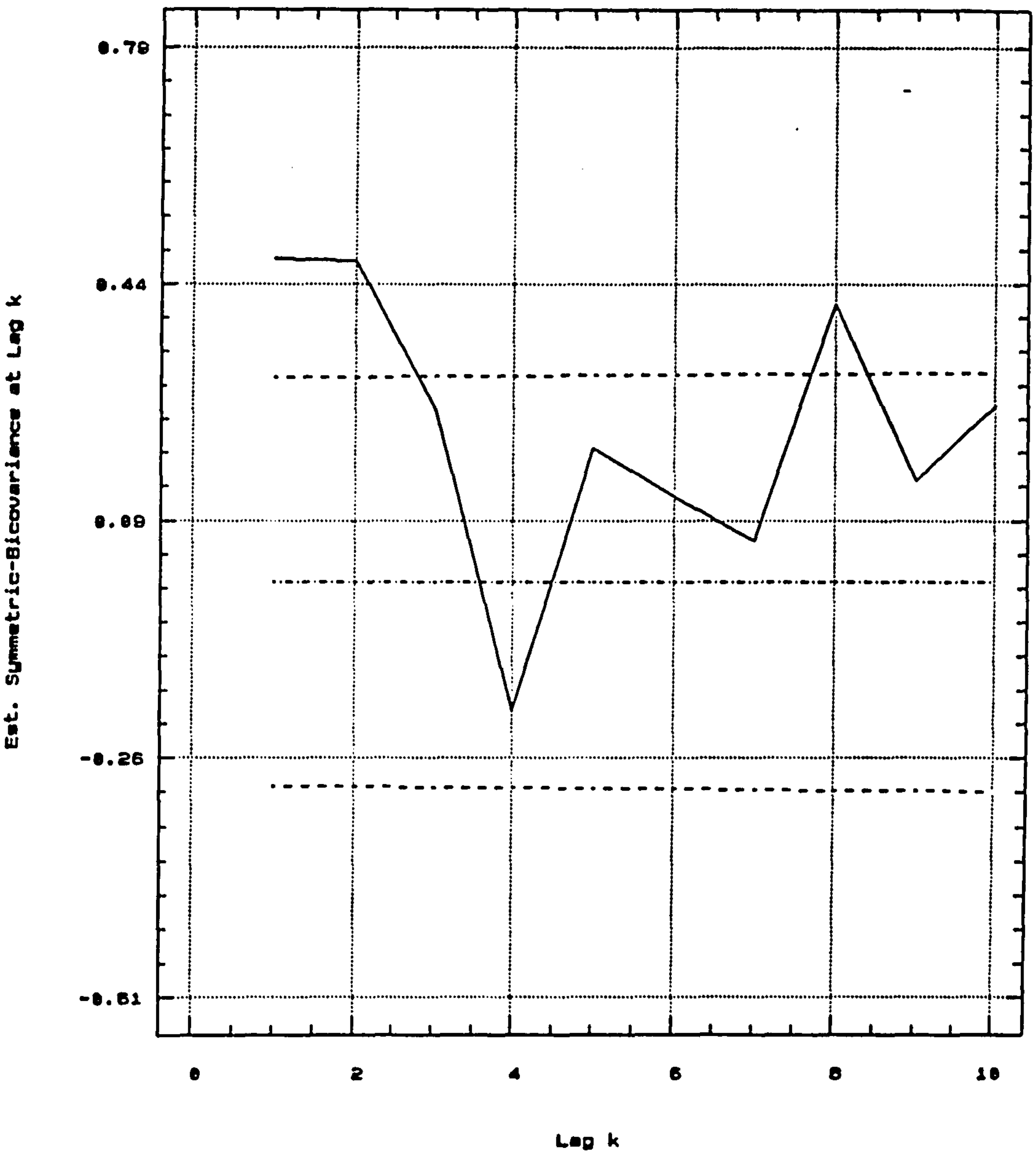


Figure 2

Est. Symmetric-Bicovariance Funct. for

ARMA(3,3) Residuals of Lynx Data

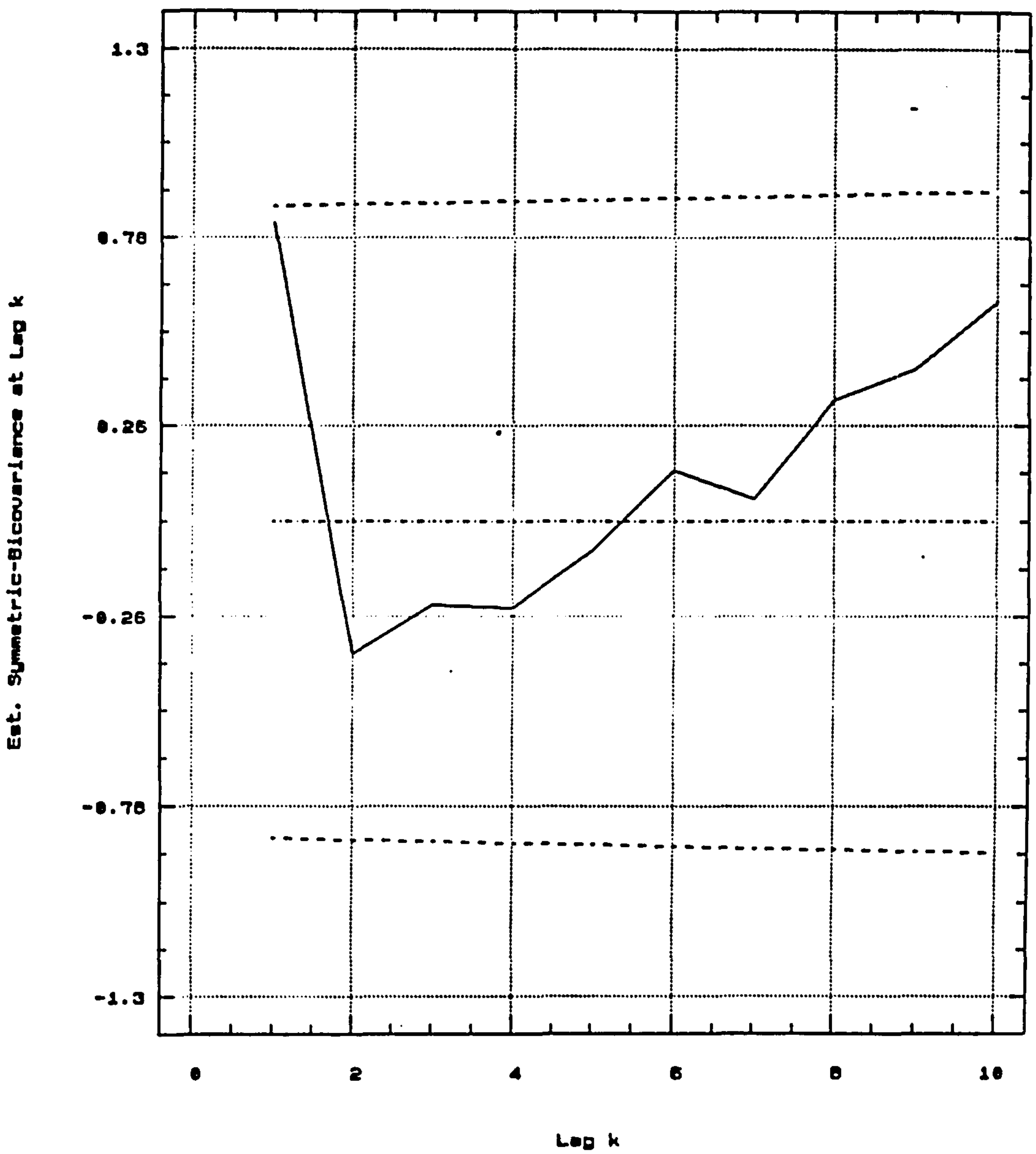
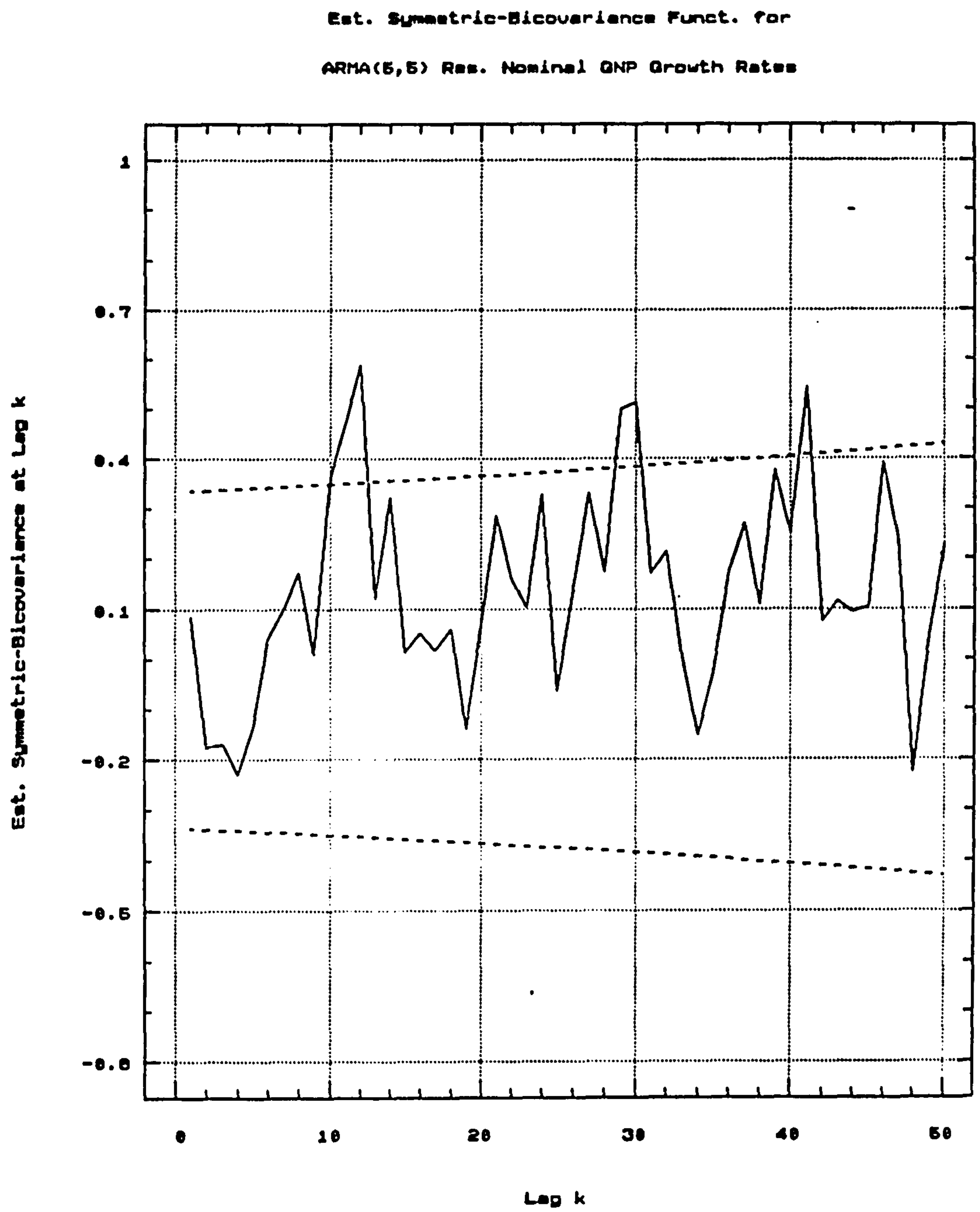


Figure 3



Est. Symmetric-Bicovariance Funct. for
ARMA(5,5) Res. Nominal GNP Growth Rates
1
0.7
0.4
0.1
-0.2
-0.5
-0.8
0
10
20
30
40
50
Lag k
Est. Symmetric-Bicovariance at Lag k

Figure 4

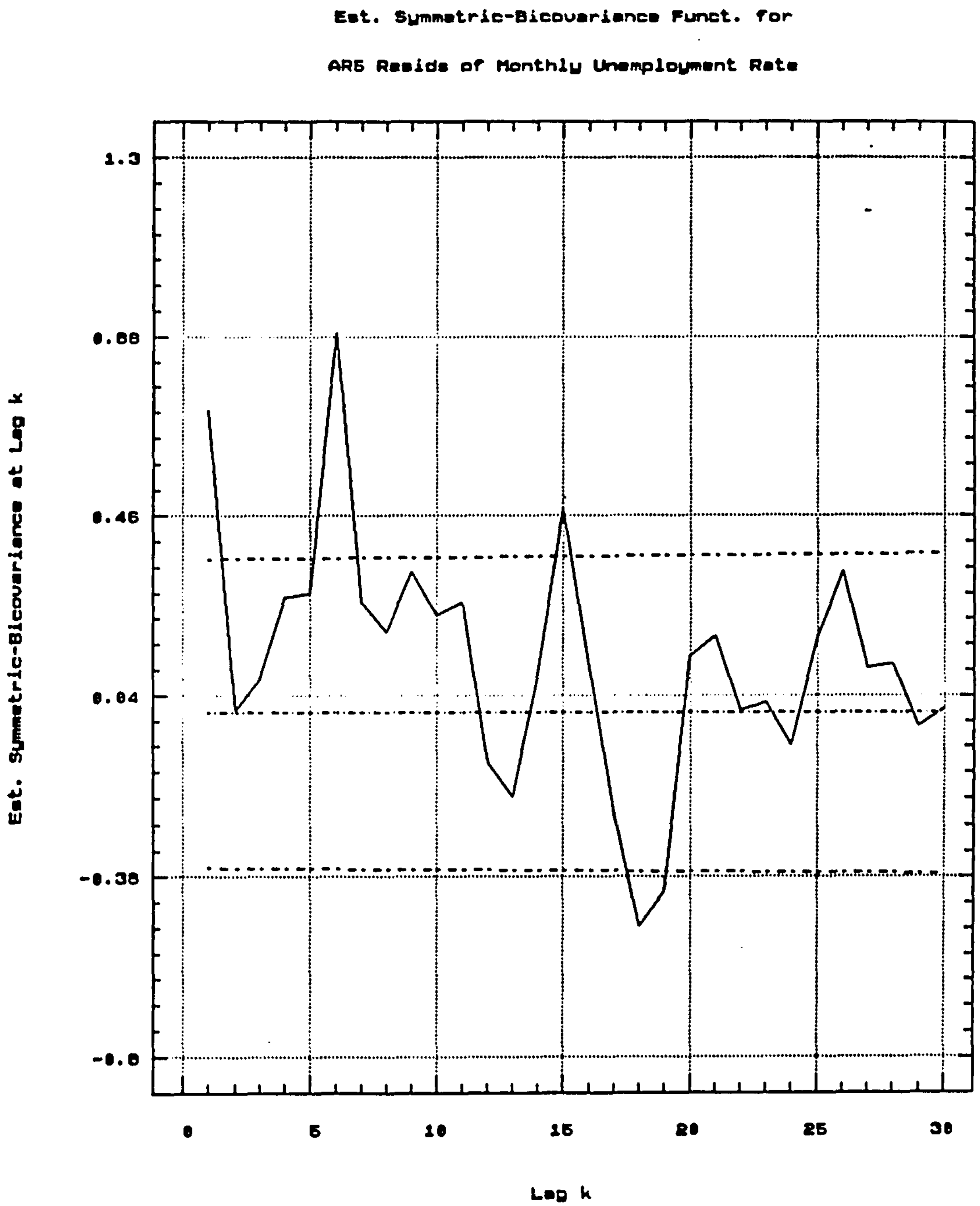

Est. Symmetric-Bicovariance Funct. for
AR5 Resids of Monthly Unemployment Rate
1.3
0.88
0.46
0.04
-0.38
-0.8
Est. Symmetric-Bicovariance at Lag k
0
5
10
15
20
25
30
Lag k

Figure 5

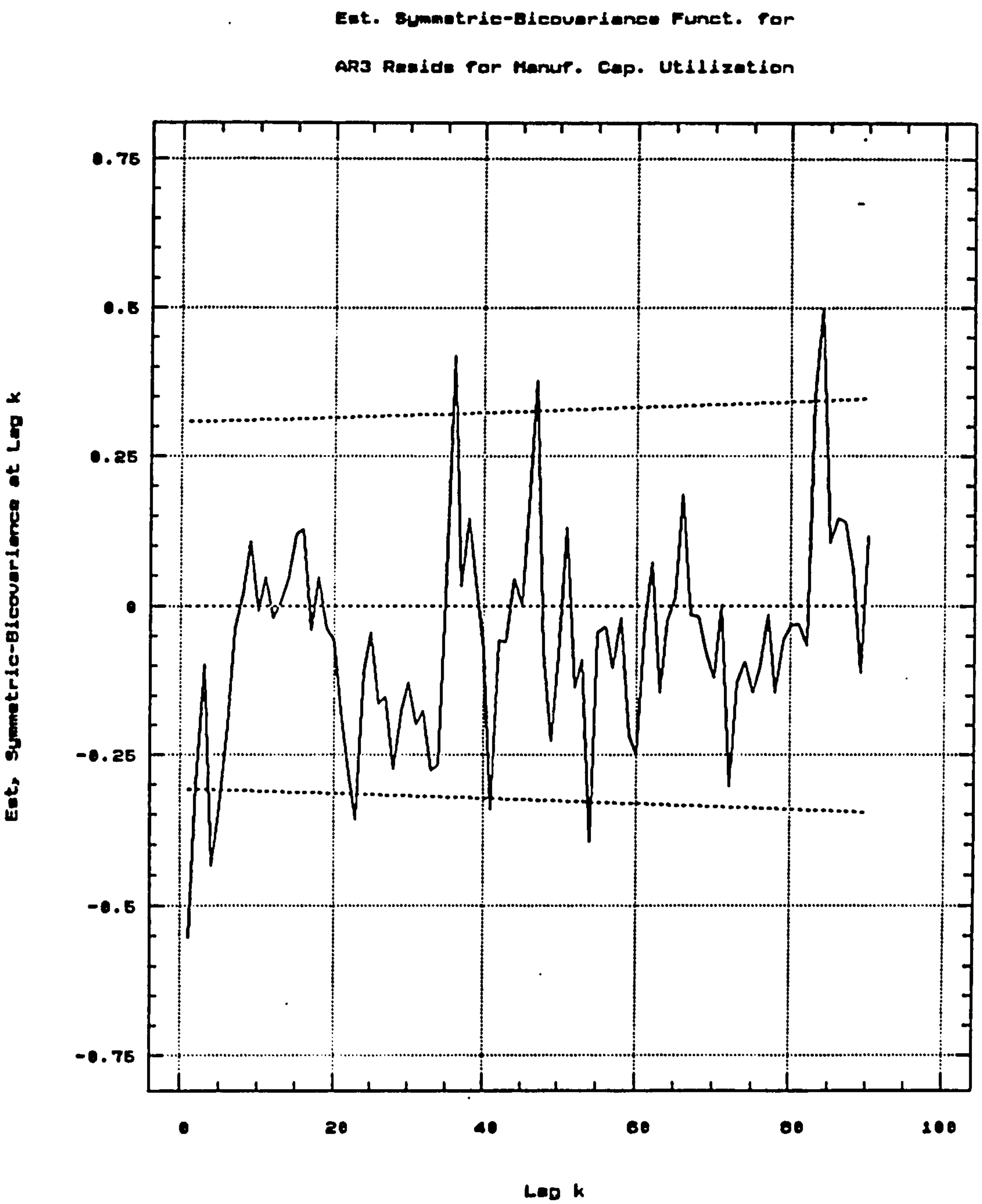

Est. Symmetric-Bicovariance Funct. for
AR3 Resids for Manuf. Cap. Utilization
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
Est, Symmetric-Bicovariance at Lag k
0
20
40
60
80
100
Lag k

Figure 6

Est. Symmetric-Bicovariance Funct. for

MA3 Resids of Pigiron Growth Rates

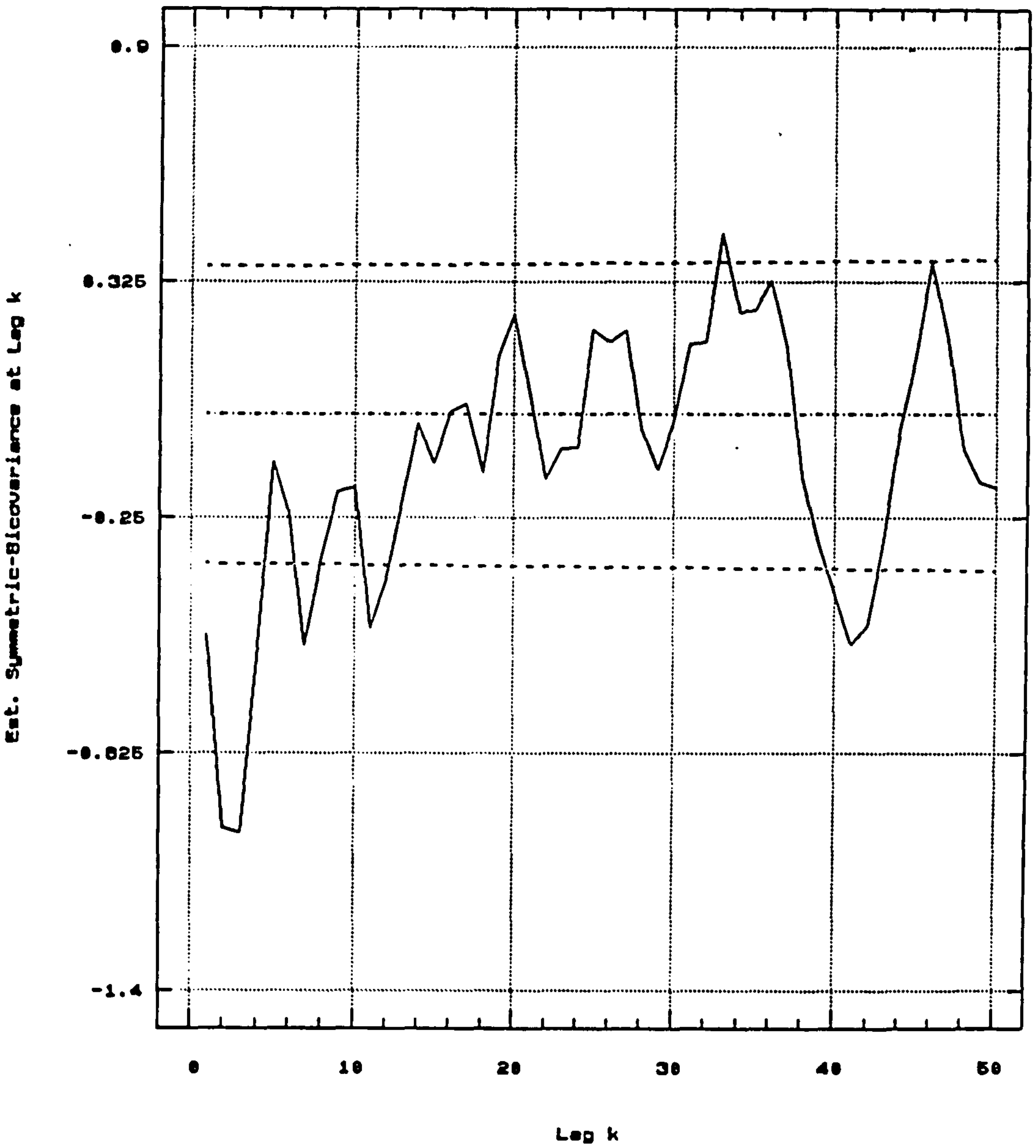


Figure 7

Est. Symmetric-Bicovariance Funct. for

Computed CRSP Stock Returns Data

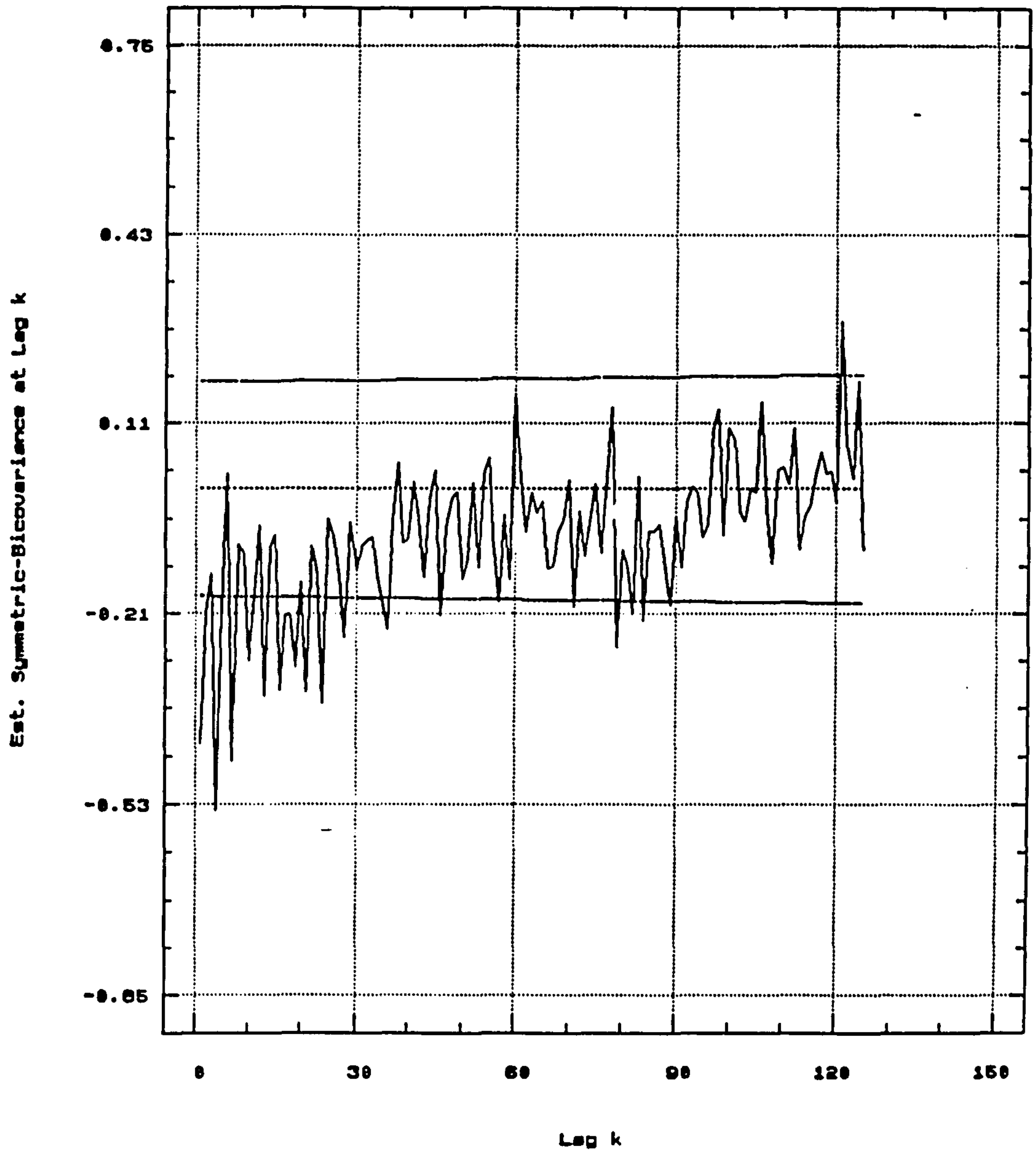


Figure 8

Est. Symmetric-Bicovariance Funct. for

AR(2) Residuals of Spot Cotton Prices